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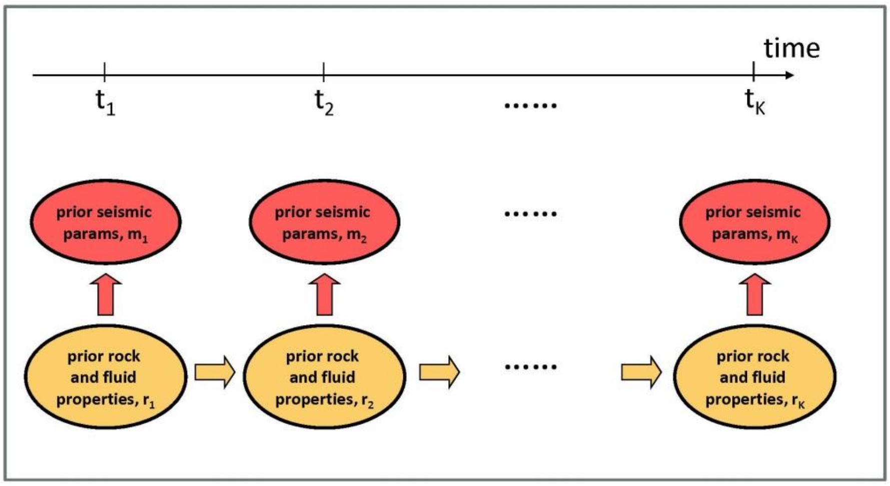
Note

# Making a 4D seismic prior model from rock physics relations

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## Abstract

This note describes how to find a prior 4D model for seismic parameters from an underlying rock physics model. The main ideas are expressed in terms of the Markov property for the dependency between different time steps, time locality in the sense that the seismic parameters at any time instance are found from the rock physics parameters as they are at that same time instance, and the use of Gaussian-linear models.



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# Contents

# 1 Introduction

A major part of the NFR funded MonCO2 project is to integrate seismic-travel time data, seismic amplitude data, and gravimetric measurements with a stochastic rock physics model in order to obtain a best possible picture of the spatial distribution of the injected $CO_2$.

This note describes how to use the underlying rock physics model to determine a 4D prior model for seismic parameters. The main 4D-framework for the inversion and summary of the workflow is documented in the note *Joint 4D inversion of multiple data sources for $CO_2$ monitoring* (Kolbjørnsen and Kjønsberg, 2011). Details regarding other individual steps are documented in separate notes. We start this note with a short review of the main ideas expressed in *Joint 4D inversion*, and then describe how to obtain the desired quantities from the rock physics model.

## 1.1 Prelude

As described in *Joint 4D inversion of multiple data sources for $CO_2$ monitoring*, the unknown parameters seen from the point of view of data inversion are a set of seismic parameters defined at each time step, $\{\boldsymbol{m}_k\}_{k=1}^K$. We use a prior model for these parameters that is a Markov chain. The Markov property is expressed as

$$p(\boldsymbol{m}_k|\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_{k-1}) = p(\boldsymbol{m}_k|\boldsymbol{m}_{k-1}),$$

and it implies that the joint distribution for all time steps can be written in terms of the probability distribution at the initial state and the transition probabilities for each successive state:

$$p(\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_K) = p(\boldsymbol{m}_1) \prod_{k=2}^K p(\boldsymbol{m}_k|\boldsymbol{m}_{k-1}).$$

This joint distribution is thus defined by the distributions:

$$p(\boldsymbol{m}_1); \quad p(\boldsymbol{m}_k|\boldsymbol{m}_{k-1}), \qquad k = 2, \dots K$$

The model we will use is formulated in a Gaussian-linear framework, in which case the statistical model for the forward transitions is described by the relations

$$\boldsymbol{m}_k = \boldsymbol{A}_k \boldsymbol{m}_{k-1} + \Delta \boldsymbol{m}_k, \qquad k = 2, \dots, K,$$

where $\boldsymbol{A}_k$ , $k = 2, \dots, K$ are matrices; and $\boldsymbol{m}_1$ and $\Delta \boldsymbol{m}_k$ , $k = 2, \dots, K$ are independent random vectors with the distributions:

$$\boldsymbol{m}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$$

$$\Delta \boldsymbol{m}_k \sim N(\Delta \boldsymbol{\mu}_k, \boldsymbol{\Delta}_k), \qquad k = 2, \dots, K$$

It follows that at any time instance $k$ the seismic parameters have a multi-normal distribution

$$\boldsymbol{m}_k \sim N(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k).$$

The mean and covariance are obtained from the recursive relations:

$$\boldsymbol{\mu}_k = \boldsymbol{A}_k \boldsymbol{\mu}_{k-1} + \Delta \boldsymbol{\mu}_k,$$

$$\boldsymbol{\Sigma}_k = \boldsymbol{A}_k \boldsymbol{\Sigma}_{k-1} \boldsymbol{A}_k{}^T + \boldsymbol{\Delta}_k.$$

Thus what is needed for the 4D prior model for the data inversion is to specify the mean and covariance of the seismic parameters at the initial time,

$$\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1;$$

the transition matrices for all later time steps,

$$\boldsymbol{A}_k, \qquad k = 2, \ldots, K;$$

and the mean and covariance of the error term for each time step,

$$\Delta\boldsymbol{\mu}_k, \boldsymbol{\Delta}_k, \qquad k = 2, \ldots, K.$$

The following sections describe how to do this from the rock physics model and statistical rock physics parameters.

# 2 Methodology

## 2.1 4D model for seismic and rock physics parameters

Figure 1 illustrates the model that relates 4D seismic parameters to rock physics relations. At each time step $k = 1, \ldots, K$ we have a set of point-wise rock physics parameters denoted $\boldsymbol{r}_k$. These are related by a Markov chain, hence the joint probability is

$$p(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_K) = p(\boldsymbol{r}_1) \prod_{k=2}^{K} p(\boldsymbol{r}_k | \boldsymbol{r}_{k-1}).$$

The relation $p(\boldsymbol{r}_k | \boldsymbol{r}_{k-1})$ is represented by the horizontal, yellow arrows in Figure 1. We assume locality in time, meaning that

$$p(\boldsymbol{m}_k | \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_K) = p(\boldsymbol{m}_k | \boldsymbol{r}_k).$$

This is represented by the vertical (red) arrows in the figure. We presume there is a well-defined rock physics model that at any time step allows us to compute the seismic parameters from the rock physics parameters:

$$\boldsymbol{m}_k = R(\boldsymbol{r}_k).$$

The rock physics relations expressed by this function can be of many kinds, for instance a differential effective medium model. At this point we do not need to specify it any further, just presume that it is known.

The Markov property in the rock physics parameters and locality in time imply a Markov property for the seismic parameters:

$$p(\boldsymbol{m}_k | \boldsymbol{m}_1, \boldsymbol{m}_2, \ldots, \boldsymbol{m}_{k-1}) = p(\boldsymbol{m}_k | \boldsymbol{m}_{k-1}).$$

This is in Figure 1 represented by the horizontal, faint red arrows. This Markov property is approximated by a linear model:

$$\boldsymbol{m}_k = \boldsymbol{A}_k \boldsymbol{m}_{k-1} + \Delta \boldsymbol{m}_k \quad k = 2, \dots, K.$$

For any given $k$ the matrix $\boldsymbol{A}_k$ is a fixed transition matrix, while the term $\Delta \boldsymbol{m}_k$ describes additional stochastic variability.

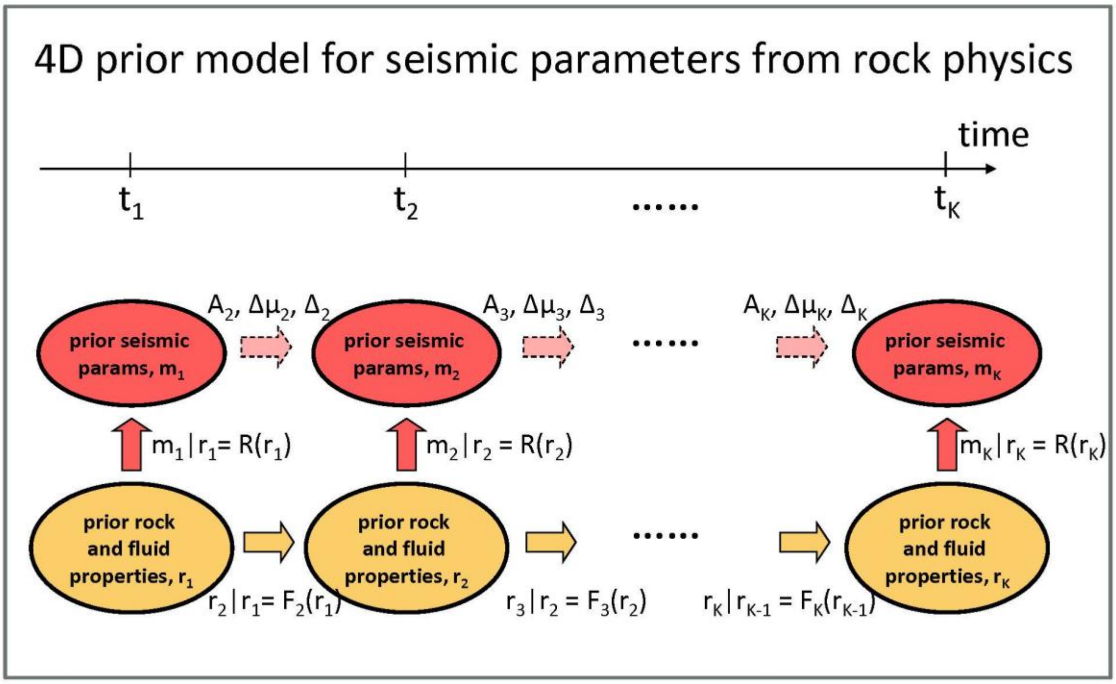


Figure 1 Seismic 4D prior from rock physics. The Markov property for rock physics parameters is shown with yellow arrows, the locality in time in the relation between seismic parameters and rock physics parameters is shown by red, vertical arrows. The implied Markov property for the seismic parameters is indicated with dashed, faint red, horizontal arrows.

## 2.2 Splitting into static and dynamic seismic parameters

Typically, the seismic parameter vector $\boldsymbol{m}$ represents the seismic wave velocities for pressure and shear waves, $V_p$ and $V_s$, in addition to the density $\rho$. That is, $\boldsymbol{m} = [V_p, V_s, \rho]^T$, a 3-dimensional vector. As discussed in *Joint 4D inversion of multiple data sources for $CO_2$ monitoring*, we choose to separately model the dynamic and static components of the seismic parameters. That is, $\boldsymbol{m} = \boldsymbol{m}_\mathrm{s} + \boldsymbol{m}_\mathrm{d}$, where the static and dynamic components of $V_p$, $V_s$, and $\rho$ are sorted into $\boldsymbol{m}_\mathrm{s}$ and $\boldsymbol{m}_\mathrm{d}$, respectively. From now we represent this as the 6-dimensional vector

$$\boldsymbol{m} = \begin{bmatrix} \boldsymbol{m}_\mathrm{s} \\ \boldsymbol{m}_\mathrm{d} \end{bmatrix}.$$

This implies that

$$\boldsymbol{\mu}_k = \begin{bmatrix} \boldsymbol{\mu}_\mathrm{s} \\ \boldsymbol{\mu}_{\mathrm{d},k} \end{bmatrix}$$

and

$$\boldsymbol{\Sigma}_k = \begin{bmatrix} \boldsymbol{\Sigma}_\mathrm{ss} & \boldsymbol{\Sigma}_{\mathrm{sd},k} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k} \end{bmatrix}.$$

The static and dynamic components of these vectors and matrices are to be found from the rock physics model. Typically, the rock physics parameters $\boldsymbol{r}_k$ can be sorted into static and dynamic parameters. But the relation that calculates seismic parameters from rock physics parameters, $\boldsymbol{m}_{\mathrm{s}}+\boldsymbol{m}_{\mathrm{d},k}=R(\boldsymbol{r}_k)$, generally does not separately compute $\boldsymbol{m}_{\mathrm{s}}$ and $\boldsymbol{m}_{\mathrm{d},k}$, only their sum. Hence there is some freedom in what we define as the static and dynamic parts of the seismic parameters. We will come back to this later.

## 2.3 Time correlated samples

For a given reservoir there will typically be some rock physics parameters for which it is unphysical to consider variations with time. These are classified as static rock physics parameters. An example of this might be, but need not be, clay density. Other parameters may clearly be time dependent. In a setting of monitoring $CO_2$ injection into an aquifer, the saturation of $CO_2$ versus brine is going to depend on time, and there may also be geochemical changes to the rock. Quite generally, we can assume that any rock physics parameter can be classified as either dynamic or static. Thus if $\boldsymbol{r}_k$ is the vector of all rock physics parameters at time instance $k$, it can be written in the separated form

$$\boldsymbol{r}_k=\begin{bmatrix}\boldsymbol{r}_{\mathrm{s}}\\ \boldsymbol{r}_{\mathrm{d},k}\end{bmatrix}.$$

The static rock physics parameters $\boldsymbol{r}_{\mathrm{s}}$ are time independent.

We can use the distinction between dynamic and static parameters to establish time correlations in a set of sampled rock physics parameters

$$\left\{\boldsymbol{r}_{k,q}=\begin{bmatrix}\boldsymbol{r}_{\mathrm{s},q}\\ \boldsymbol{r}_{\mathrm{d},k,q}\end{bmatrix}\right\}_{k=1}^{K}.$$

Here $q\in\{1,2,\dots,Q\}$ labels the set, and at each time instance $k$ we have a sample of each rock physics parameter. Time correlations can be obtained as follows: At the initial time instance, $k=1$, sample all the static parameters $\boldsymbol{r}_{\mathrm{s},q}$ (they are static, but can very well be stochastic), and at the time steps $k=1,2,\dots,K$ also sample the dynamic parameters. The latter can be sampled independently at each time step, although a physically more feasible way is to sample the conditioned parameter $\boldsymbol{r}_{\mathrm{d},k,q}|\boldsymbol{r}_{\mathrm{d},k-1,q}$. The set

$$\boldsymbol{r}_{\mathrm{s},q},\quad \boldsymbol{r}_{\mathrm{d},1,q},\quad \boldsymbol{r}_{\mathrm{d},2,q}|\boldsymbol{r}_{\mathrm{d},1,s},\dots,\quad \boldsymbol{r}_{\mathrm{d},K,q}|\boldsymbol{r}_{\mathrm{d},K-1,q}$$

then has time correlations arising from two sources: 1) the use of the same sampled values for the static parameters at all time instances; 2) the conditioning done when sampling the dynamic parameters. For given $k$ and $q$ compute the seismic parameters according to

$$\boldsymbol{m}_{\mathrm{s},q}+\boldsymbol{m}_{\mathrm{d},k,q}=R\big(\boldsymbol{r}_{\mathrm{s},q},\,\boldsymbol{r}_{\mathrm{d},k,q}|\boldsymbol{r}_{\mathrm{d},k-1,q}\big).$$

If no static rock physics parameters exist, time correlations *must* be obtained through conditioned sampling of $\boldsymbol{r}_{\mathrm{d},k,q}|\boldsymbol{r}_{\mathrm{d},k-1,q}$ in order to obtain time correlations. This relation typically will be of a stochastic nature.

Doing this for $q=1,\dots,Q$ we obtain the matrix

$$\begin{bmatrix}\boldsymbol{m}_{\mathrm{s},1}+\boldsymbol{m}_{\mathrm{d},1,1} & \cdots & \boldsymbol{m}_{\mathrm{s},Q}+\boldsymbol{m}_{\mathrm{d},1,Q}\\ \vdots & \ddots & \vdots\\ \boldsymbol{m}_{\mathrm{s},1}+\boldsymbol{m}_{\mathrm{d},K,1} & \cdots & \boldsymbol{m}_{\mathrm{s},Q}+\boldsymbol{m}_{\mathrm{d},K,Q}\end{bmatrix},$$

where each column represents a set of time correlated seismic parameters. Each row represents a series of independent samples for a given time. Two neighboring rows have pairwise (column-wise) time correlated samples. For instance is $\boldsymbol{m}_{\mathrm{s},q} + \boldsymbol{m}_{\mathrm{d},k,q}$ correlated with $\boldsymbol{m}_{\mathrm{s},q} + \boldsymbol{m}_{\mathrm{d},k-1,q}$. The sampled values of this matrix will in the following three sections be used to estimate to desired quantities

$$\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1; \qquad \boldsymbol{A}_k, \Delta\boldsymbol{\mu}_k, \boldsymbol{\Delta}_k, \qquad k = 2, \ldots, K.$$

## 2.4 Point-wise prior for seismic parameters at time t = 0

In section 1.1 we saw that it is necessary to establish the mean and covariance that describe the Gaussian distribution at the first time, $k = 1$. That is, we need to establish

$$\boldsymbol{\mu}_1 = \begin{bmatrix} \boldsymbol{\mu}_\mathrm{s} \\ \boldsymbol{\mu}_{\mathrm{d},1} \end{bmatrix}, \boldsymbol{\Sigma}_1 = \begin{bmatrix} \boldsymbol{\Sigma}_\mathrm{ss} & \boldsymbol{\Sigma}_{\mathrm{sd},1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},1} & \boldsymbol{\Sigma}_{\mathrm{dd},1} \end{bmatrix},$$

these being the parameters that define the distribution function for the seismic parameters at time t = 0 ,

$$\boldsymbol{m}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1).$$

In our model the same distribution function is used for all cells of the gridded reservoir. Any spatial trends in the seismic parameters are treated separately, and will be discussed in section 3.1. Spatial correlations between the seismic parameters are also handled separately. More on this can be found in Buland et al., 2003.

We define the seismic parameters for the first time instance, t = 0 ($k = 1$), to be all static. That is,

$$\boldsymbol{\mu}_1 = \begin{bmatrix} \boldsymbol{\mu}_\mathrm{s} \\ \boldsymbol{0} \end{bmatrix}, \boldsymbol{\Sigma}_1 = \begin{bmatrix} \boldsymbol{\Sigma}_\mathrm{ss} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

For the matrix of correlated samples from section 2.3, this means that the dynamic part of each sample for $k = 1$ is zero, i.e. $\boldsymbol{m}_{\mathrm{d},1,q} = \boldsymbol{0}$ for all $q = 1, \ldots, Q$. Estimation of the mean and covariance is then done by

$$\boldsymbol{\mu}_\mathrm{s} = \frac{1}{Q} \sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{s},q},$$

$$\boldsymbol{\Sigma}_\mathrm{ss} = \mathrm{Cov}(\boldsymbol{m}_\mathrm{s}, \boldsymbol{m}_\mathrm{s}) = \frac{1}{Q} \sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{s},q} \boldsymbol{m}_{\mathrm{s},q}^T - \boldsymbol{\mu}_\mathrm{s} \boldsymbol{\mu}_\mathrm{s}^T.$$

## 2.5 Transition matrices

The relation $\boldsymbol{m}_k = \boldsymbol{A}_k \boldsymbol{m}_{k-1} + \Delta\boldsymbol{m}_k$ implies that

$$\boldsymbol{A}_k = \mathrm{Cov}(\boldsymbol{m}_k, \boldsymbol{m}_{k-1}) \boldsymbol{\Sigma}_{k-1}^{-1}; \qquad \boldsymbol{\Sigma}_{k-1} = \mathrm{Cov}(\boldsymbol{m}_{k-1}, \boldsymbol{m}_{k-1}).$$

We start by estimating

$$\boldsymbol{\mu}_k = \begin{bmatrix} \boldsymbol{\mu}_\mathrm{s} \\ \boldsymbol{\mu}_{\mathrm{d},k} \end{bmatrix}, \boldsymbol{\Sigma}_k = \begin{bmatrix} \boldsymbol{\Sigma}_\mathrm{ss} & \boldsymbol{\Sigma}_{\mathrm{sd},k} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k} \end{bmatrix},$$

and treat the static and dynamic parts separately, for clarity. Our choice of defining the seismic parameters for the first time instance to be all static, means that for t = 0 the dynamic part of the matrix of correlated samples in section 2.3, is zero,

$$\boldsymbol{m}_{\mathrm{d},1,q} = \boldsymbol{0},$$

and for any sample $\boldsymbol{m}_{k,q} = \boldsymbol{m}_{\mathrm{s},q} + \boldsymbol{m}_{\mathrm{d},k,q}$ , $k > 1$, we identify the dynamic parts as the difference between this sample and the corresponding (same $q$) sample at t = 0:

$$\boldsymbol{m}_{\mathrm{d},k,q} = \boldsymbol{m}_{k,q} - \boldsymbol{m}_{1,q} = \boldsymbol{m}_{k,q} - \boldsymbol{m}_{\mathrm{s},q}.$$

Estimation of the mean and covariance elements is then:

$$\boldsymbol{\mu}_{\mathrm{d},k} = \frac{1}{Q}\sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{d},k,q},$$

$$\boldsymbol{\Sigma}_{\mathrm{sd},k} = \mathrm{Cov}\left(\boldsymbol{m}_{\mathrm{s}}, \boldsymbol{m}_{\mathrm{d},k}\right) = \frac{1}{Q}\sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{s},q}\boldsymbol{m}_{\mathrm{d},k,q}^{T} - \boldsymbol{\mu}_{\mathrm{s}}\boldsymbol{\mu}_{\mathrm{d},k}^{T},$$

$$\boldsymbol{\Sigma}_{\mathrm{ds},k} = \boldsymbol{\Sigma}_{\mathrm{sd}}^{T},$$

$$\boldsymbol{\Sigma}_{\mathrm{dd},k} = \mathrm{Cov}\left(\boldsymbol{m}_{\mathrm{d},k}, \boldsymbol{m}_{\mathrm{d},k}\right) = \frac{1}{Q}\sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{d},k,q}\boldsymbol{m}_{\mathrm{d},k,q}^{T} - \boldsymbol{\mu}_{\mathrm{d},k}\boldsymbol{\mu}_{\mathrm{d},k}^{T}.$$

We can furthermore write the covariance between two consecutive time instances as

$$\boldsymbol{D}_{k,k-1} \equiv \mathrm{Cov}(\boldsymbol{m}_{k}, \boldsymbol{m}_{k-1}) = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k-1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1} \end{bmatrix}.$$

with

$$\boldsymbol{\Sigma}_{\mathrm{dd},k,k-1} \equiv \mathrm{Cov}\left(\boldsymbol{m}_{\mathrm{d},k}, \boldsymbol{m}_{\mathrm{d},k-1}\right) = \frac{1}{Q}\sum_{q=1}^{Q} \boldsymbol{m}_{\mathrm{d},k,q}\boldsymbol{m}_{\mathrm{d},k-1,q}^{T} - \boldsymbol{\mu}_{\mathrm{d},k}\boldsymbol{\mu}_{\mathrm{d},k-1}^{T}.$$

Now all elements needed for finding the transition matrices are estimated from rock physics parameters, and the calculation of the matrices $\boldsymbol{A}_k$ for $k = 2, \dots, K$ is straightforward:

$$\boldsymbol{A}_{k} = \boldsymbol{D}_{k,k-1}\boldsymbol{\Sigma}_{k-1}^{-1}.$$

In the appendix of section 5.1 we prove that the matrix $\boldsymbol{A}_k$ is on the form

$$\boldsymbol{A}_{k} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{\mathrm{sd},k} & \boldsymbol{A}_{\mathrm{dd},k} \end{bmatrix}.$$

This means that

$$\boldsymbol{A}_k\boldsymbol{m}_{k-1} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{\mathrm{sd},k} & \boldsymbol{A}_{\mathrm{dd},k} \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_{\mathrm{s}} \\ \boldsymbol{m}_{\mathrm{d},k-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{m}_{\mathrm{s}} \\ \boldsymbol{A}_{\mathrm{sd},k}\boldsymbol{m}_{\mathrm{s}} + \boldsymbol{A}_{\mathrm{dd},k}\boldsymbol{m}_{\mathrm{d},k-1} \end{bmatrix}.$$

The static component of the seismic parameters is unchanged, but the new dynamic part gets contributions from both the static and dynamic part of the seismic parameter $\boldsymbol{m}_{k-1}$.

## 2.6 Correction term

What is left now is to establish the mean and covariance of the correction term $\Delta\boldsymbol{m}_k$, that is to find

$$\Delta\boldsymbol{\mu}_k, \boldsymbol{\Delta}_k.$$

The expression $\boldsymbol{m}_k = \boldsymbol{A}_k\boldsymbol{m}_{k-1} + \Delta\boldsymbol{m}_k$ implies that $\Delta\boldsymbol{m}_k = \boldsymbol{m}_k - \boldsymbol{A}_k\boldsymbol{m}_{k-1}$. Hence

$$\begin{aligned} &\mathrm{Cov}(\Delta\boldsymbol{m}_k, \Delta\boldsymbol{m}_k) \\ &\quad = \mathrm{Cov}(\boldsymbol{m}_k, \boldsymbol{m}_k) - \mathrm{Cov}(\boldsymbol{m}_k, \boldsymbol{m}_{k-1})\boldsymbol{A}_k^T - \boldsymbol{A}_k\mathrm{Cov}(\boldsymbol{m}_{k-1}, \boldsymbol{m}_k) \\ &\quad + \boldsymbol{A}_k\mathrm{Cov}(\boldsymbol{m}_{k-1}, \boldsymbol{m}_{k-1})\boldsymbol{A}_k^T \\ &\quad = \boldsymbol{\Sigma}_k - \boldsymbol{D}_{k,k-1}\left(\boldsymbol{\Sigma}_{k-1}^{-1}\right)^T\boldsymbol{D}_{k,k-1}^T - \boldsymbol{D}_{k,k-1}\boldsymbol{\Sigma}_{k-1}^{-1}\boldsymbol{D}_{k,k-1}^T \\ &\quad + \boldsymbol{D}_{k,k-1}\boldsymbol{\Sigma}_{k-1}^{-1}\boldsymbol{\Sigma}_{k-1}\left(\boldsymbol{\Sigma}_{k-1}^{-1}\right)^T\boldsymbol{D}_{k,k-1}^T = \boldsymbol{\Sigma}_k - \boldsymbol{D}_{k,k-1}\boldsymbol{A}_k^T. \end{aligned}$$

That is, with the estimations of the previous section we also have estimated

$$\boldsymbol{\Delta}_k = \mathrm{Cov}(\Delta\boldsymbol{m}_k, \Delta\boldsymbol{m}_k) = \boldsymbol{\Sigma}_k - \boldsymbol{D}_{k,k-1}\boldsymbol{A}_k^T = \boldsymbol{\Sigma}_k - \boldsymbol{D}_{k,k-1}\left(\boldsymbol{\Sigma}_{k-1}^{-1}\right)^T\boldsymbol{D}_{k,k-1}^T.$$

In addition to the covariance between the two time steps, held by the matrix $\boldsymbol{D}_{k,k-1}$, the covariance of the correction term includes the covariance matrices for both time steps separately. In the appendix of section 5.2 we prove that the matrix $\boldsymbol{\Delta}_k$ is on the form

$$\boldsymbol{\Delta}_k = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Delta}_{\mathrm{dd},k} \end{bmatrix}.$$

We can also find the mean of the correction term:

$$\Delta\boldsymbol{\mu}_k = \boldsymbol{\mu}_k - \boldsymbol{A}_k\boldsymbol{\mu}_{k-1} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\mu}_{\mathrm{d},k} - \boldsymbol{A}_{\mathrm{sd},k}\boldsymbol{\mu}_{\mathrm{s}} - \boldsymbol{A}_{\mathrm{dd},k}\boldsymbol{\mu}_{\mathrm{d},k-1} \end{bmatrix}.$$

The correction term, as estimated by means of the formula $\Delta\boldsymbol{m}_k = \boldsymbol{m}_k - \boldsymbol{A}_k\boldsymbol{m}_{k-1}$, is thus concerned only with the dynamic part of the seismic parameters, not the static part.

In addition to the contributions from the rock physics model, as computed above, we might also let $\Delta\boldsymbol{m}_k$ contain an element independent of the rock physics. This would then represent an independent error term. There is little reason to let this error provide any bias or additional correlations to $\Delta\boldsymbol{m}_k$. Hence we would model it in terms of

$$\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma^2\boldsymbol{I}),$$

alternatively with separate variance for the static and the dynamic part of the seismic parameters.

Let us also mention that it would be possible to use a slightly different model. If we insist that the correction term has zero mean, i.e. does not itself change the mean value of the seismic parameter, it is necessary to make up for this by a changed covariance. This can be achieved by a model $\boldsymbol{m}_k = \boldsymbol{A}_k \boldsymbol{m}_{k-1} + \Delta\widetilde{\boldsymbol{m}}_k$, where the new correction term is normally distributed with mean and covariance given by

$$\Delta\widetilde{\boldsymbol{\mu}}_k = \mathbf{0}, \qquad \widetilde{\boldsymbol{\Delta}}_k = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_{\mathrm{dd},k} + \Delta\boldsymbol{\mu}_{\mathrm{d},k}(\Delta\boldsymbol{\mu}_{\mathrm{d},k})^T \end{bmatrix}.$$

# 3 Special issues

## 3.1 Trends

As notes above, our model for seismic parameters, as built from the underlying rock physics model, is stationary. That is, the same distribution function is used for all cells of the gridded reservoir. This might be an unphysical assumption, as it is not uncommon to have for instance depth trends in temperature and pressure. In the case of $CO_2$ injection there might also be horizontal trends. The $CO_2$ will, because of buoyancy, naturally move upwards, but to some extent some horizontal movement will also occur. The latter is enhanced by shale layers and other obstacles hindering the vertical flow. At Utsira there are publications reporting on a (horizontal) temperature difference of 10°C from the central plume to its outskirts.

It is important to find out to which extent such non-stationary effects should be included in our model. It seems clear that if for instance a temperature gradient is to be included, the reason must be that it has a significant effect on what can be detected. I first analysis of this is to study to which extent the gradient has any effect on the seismic parameters as predicted by the forward rock physics models. The sensitivity to parameter changes will typically differ among rock physics models. A second question is to which extent the data inversion will be able to distinguish between the seismic parameters obtained by the various, say, temperatures.

The physical existence of trends should be studied for each storage site and injection case. Subsequently the need to take into account these trends, if present, in the model should be investigated for each rock physics model.

If the outcome of such an analysis is that trend(s) should be included, this needs to be handled outside the prior model building described in these notes. As long as the trends do not change the correlations of the seismic parameters, the effects of any trend can be added to the mean value $\boldsymbol{\mu}_1$.

# 4 References

O. Kolbjørnsen and H. Kjønsberg. *Joint 4D inversion of multiple data sources for CO2 monitoring*. NR Note, SAND/20/2011, December 2011.

A. Buland, O. Kolbjørnsen, and H. Omre. *Rapid spatially coupled AVO inversion in the Fourier domain*. Geophysics **68** (3), 2003.

# 5 Appendix: Technical issues

## 5.1 Proof for the form of the transition matrices

We prove here that the transition matrices have the form

$$\boldsymbol{A}_k = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{\mathrm{sd},k} & \boldsymbol{A}_{\mathrm{dd},k} \end{bmatrix}.$$

Using the relations of section 2.5 , and the short-hand notation

$$\boldsymbol{F} = \left(\boldsymbol{\Sigma}_{\mathrm{dd},k-1} - \boldsymbol{\Sigma}_{\mathrm{ds},k-1}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}\boldsymbol{\Sigma}_{\mathrm{sd},k-1}\right)^{-1},$$

we have:

$$\begin{aligned}
\boldsymbol{A}_k = \boldsymbol{D}_{k,k-1}\boldsymbol{\Sigma}_{k-1}^{-1} &= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k-1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k-1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k-1} & \boldsymbol{\Sigma}_{\mathrm{dd},k-1} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k-1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}(\boldsymbol{I} + \boldsymbol{\Sigma}_{\mathrm{sd},k-1}\boldsymbol{F}\boldsymbol{\Sigma}_{\mathrm{ds},k-1}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}) & -\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}\boldsymbol{\Sigma}_{\mathrm{sd},k-1}\boldsymbol{F} \\ -\boldsymbol{F}\boldsymbol{\Sigma}_{\mathrm{ds},k-1}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1} & \boldsymbol{F} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{A}_{\mathrm{sd},k} & \boldsymbol{A}_{\mathrm{dd},k} \end{bmatrix},
\end{aligned}$$

with

$$\boldsymbol{A}_{\mathrm{sd},k} = \boldsymbol{\Sigma}_{\mathrm{ds},k}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}(\boldsymbol{I} + \boldsymbol{\Sigma}_{\mathrm{sd},k-1}\boldsymbol{F}\boldsymbol{\Sigma}_{\mathrm{ds},k-1}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}) - \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1}\boldsymbol{F}\boldsymbol{\Sigma}_{\mathrm{ds},k-1}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1},$$

$$\boldsymbol{A}_{\mathrm{dd},k} = \boldsymbol{\Sigma}_{\mathrm{ds},k}\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}\boldsymbol{\Sigma}_{\mathrm{sd},k-1}\boldsymbol{F} - \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1}\boldsymbol{F}.$$

## 5.2 Proof for the form of the correction term covariance

We prove here that the correction term covariance matrices have the form

$$\boldsymbol{\Delta}_k = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Delta}_{\mathrm{dd},k} \end{bmatrix}.$$

Using the relations of section 2.5 and 2.6 we have

$$\begin{aligned}
\boldsymbol{\Delta}_k = \boldsymbol{\Sigma}_k - \boldsymbol{D}_{k,k-1}\boldsymbol{A}_k^T &= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k-1} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_{\mathrm{sd},k}^T \\ \boldsymbol{0} & \boldsymbol{A}_{\mathrm{dd},k}^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{sd},k} \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{dd},k} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{ss}} & \boldsymbol{\Sigma}_{\mathrm{ss}}\boldsymbol{A}_{\mathrm{sd},k}^T + \boldsymbol{\Sigma}_{\mathrm{sd},k-1}\boldsymbol{A}_{\mathrm{dd},k}^T \\ \boldsymbol{\Sigma}_{\mathrm{ds},k} & \boldsymbol{\Sigma}_{\mathrm{ds},k}\boldsymbol{A}_{\mathrm{sd},k}^T + \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1}\boldsymbol{A}_{\mathrm{dd},k}^T \end{bmatrix}.
\end{aligned}$$

Since

$$\boldsymbol{A}_{\mathrm{sd},k} = \left(\boldsymbol{\Sigma}_{\mathrm{ds},k} - \boldsymbol{A}_{\mathrm{dd},k}\boldsymbol{\Sigma}_{\mathrm{ds},k-1}\right)\boldsymbol{\Sigma}_{\mathrm{ss}}^{-1}$$

We find

$$\boldsymbol{\Delta}_k = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{\mathrm{ds},k}\boldsymbol{A}_{\mathrm{sd},k}^T + \boldsymbol{\Sigma}_{\mathrm{dd},k,k-1}\boldsymbol{A}_{\mathrm{dd},k}^T \end{bmatrix}.$$