A penalty scheme for solving American option problems

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Outline

- Options?
- American put, model problem.
- Penalty method.
- Numerical experiments.
- Multi-asset options.
- Conclusion.
Options?

- Option, a contract that gives the buyer the right (but no obligation) to buy (sell) an asset for a prescribed price at a prescribed expire date.
- European, American, Asian, Exotic, Barrier and Multi-asset options.
- European, exercise only permitted at expire.
- American, exercise permitted at any time during the life of the option.
Options?, continued ...

- No arbitrage, risk-free interest rate, continuous trading, etc.

- Black-Scholes equation

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} - rP = 0.
\]

- \( P = P(S, t) \); risk-neutral price of the option.
- \( S \); underlying asset.
- \( r \); interest rate.
- \( \sigma \); volatility.

- European, fixed solution domain.
- American, moving boundary.
American put

- For $S > \bar{S}(t)$ and $0 \leq t < T$
  \[
  \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} - r P = 0.
  \]
- For $0 \leq S < \bar{S}(t)$
  \[P(S, t) = E - S.\]
- $\bar{S}(t)$; unknown moving boundary.
American put, continued ...

- $E$; Exercise price.
- No arbitrage, constraint
  \[
  P(S, t) \geq \max(E - S, 0).
  \]
- $P$, $\frac{\partial P}{\partial S}$ continuous.
- Final conditions (backwards in time!)
  \[
  P(S, T) = \max(E - S, 0),
  \bar{S}(T) = E.
  \]
Penalty method

- Recall the constraint
  \[ P(S, t) \geq \max(E - S, 0). \]

- Zvan, Forsyth and Vetzal (1998);
  - Discrete \( P \) gets close to the constraint.
  - Add a “LARGE” number to the discrete equations.
  - “Push” the approx. solution away from the constraint.

- Our approach; Add a continuous penalty term to the Black-Scholes equation.
Penalty method, continued ...

- For $S \geq 0$ and $t \in [0, T)$
\[
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP + \frac{\epsilon C'}{P + \epsilon - (E - S)} = 0.
\]

- $C \geq rE$ positive constant.

- $0 < \epsilon \ll 1$.

- Nonlinear PDE posed on a fixed domain.
Penalty method, continued ...

The penalty term

\[
\frac{\epsilon C'}{P + \epsilon - (E - S)}
\]

is

• of order $\epsilon$ if $P \gg (E - S)$.

• $\approx C' \geq rE$ as $P \to (E - S)$. 
Penalty method, continued ...

- Explicit scheme (backwards in time!);

\[
\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2} \sigma^2 S_j^2 \frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{(\Delta S)^2} + r S_j \frac{P_{j+1}^n - P_j^n}{\Delta S} - r P_j^n + \frac{\epsilon C}{P_j^n + \epsilon - q_j} = 0.
\]

- Theorem 1

For all \( C \geq rE \),

\[
P_j^n \geq \max(E - S_j, 0),
\]

provided that

\[
\Delta t \leq \frac{(\Delta S)^2}{\sigma^2 S_\infty^2 + r S_\infty(\Delta S) + r(\Delta S)^2 + \frac{C}{\epsilon}(\Delta S)^2}.
\]
Penalty method, continued ...

- Fully-implicit (nonlinear equations);

\[
\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2} \sigma^2 S_j^2 \frac{P_{j-1}^{n-1} - 2P_j^{n-1} + P_{j+1}^{n-1}}{(\Delta S)^2} \\
+ r S_j \frac{P_{j+1}^{n-1} - P_j^{n-1}}{\Delta S} - r P_j^{n-1} + \frac{\epsilon C}{P_j^{n-1} + \epsilon - q_j} = 0.
\]

- Theorem 2

For all \( C \geq rE \),

\[
P_j^n \geq \max(E - S_j, 0).
\]

- No condition on \( \Delta t \) required!
Penalty method, continued ...

- Semi-implicit (linear equations);

\[
\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2} \sigma^2 S_j^2 \frac{P_{j-1}^{n-1} - 2P_j^{n-1} + P_{j+1}^{n-1}}{(\Delta S)^2} \\
+ r S_j \frac{P_{j+1}^{n-1} - P_j^{n-1}}{\Delta S} - r P_j^{n-1} = -\frac{\epsilon C}{P_j^n + \epsilon - q_j}.
\]

- **Theorem 3**

  For all \( C \geq rE \),

  \[ P_j^n \geq \max(E - S_j, 0), \]

  provided that

  \[ \Delta t \leq \frac{\epsilon}{rE}. \]
Numerical experiments

- Model parameters:
  \[ r = 0.1, \]
  \[ \sigma = 0.2, \]
  \[ E = 1, \]
  \[ T = 1. \]

- Reference solution; Implicit Front-Fixing.
Numerical experiments, continued ...

- Explicit;

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- Fully-implicit;

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- Semi-implicit;

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Multi-asset options

- Assets; $S_1, S_2$.
- Option price; $P = P(S_1, S_2, t)$.
- Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 P}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 P}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 P}{\partial S_2 \partial S_1} + r S_1 \frac{\partial P}{\partial S_1} + r S_2 \frac{\partial P}{\partial S_2} - r P = 0.$$

- $\rho$: correlation between the assets.
Multi-asset options, continued ...

- Payoff function at expire
  \[ \phi(S_1, S_2) = \max \left( E - (\alpha_1 S_1 + \alpha_1 S_1), 0 \right). \]

- American options → constraint
  \[ P(S_1, S_2, t) \geq \phi(S_1, S_2). \]

- Penalty term
  \[ \frac{\epsilon C}{P + \epsilon - (E - (\alpha_1 S_1 + \alpha_1 S_1))}. \]

- Analysis,
  \[ C \geq rE. \]
Multi-asset options, continued ...

• We define explicit, fully-implicit and semi-implicit schemes.

• $\rho = 0$, i.e. independent assets, we prove that the constraint is fulfilled.

• $\rho \neq 0$, numerical experiments indicate that the constraint is satisfied.

• Fine meshes, the semi-implicit scheme is preferable.
Conclusion

- Both American single- and multi- asset options can be priced efficiently by penalty methods.
- Explicit scheme; easy to implement, inefficient.
- Fully-implicit scheme; “hard to implement”, efficient.
- Semi-implicit; “easy to implement”, efficient.