A Note on Two–Dimensional Non-Divergent Gaussian Random Vector Fields

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Abstract

The second order properties of a two–dimensional non–divergent Gaussian random vector field are reviewed. The approach is based on a random scalar stream function, thereby avoiding the consideration of tensor properties common in the literature of turbulence theory. Thus it may be well suited as an introduction to random vector fields for researchers working with geostatistics and applied statistics.

Key word index: homogeneous and isotropic random field, stream function, non-divergent velocity field

1 Introduction

This report was prepared as part of a case study within the project "Toolkits in Industrial Mathematics", funded by the Research Council of Norway. The purpose of this report is to review the second–order properties of a two–dimensional (2–D) non–divergent Gaussian random vector field. A non–divergent vector field in two–dimensions (2–D) can be effectively described in terms of an underlying scalar *stream function.* This makes the 2–D case particularly easy to work with, thus making it a simple starting point for understanding random vector fields.

The motivation for the present work was to find a realistic class of random velocity fields to be used as input for a stochastic advection–diffusion equation. Random velocity fields occur in applications such as statistical modeling of wind velocity, ocean currents and flow in porous media. These velocity fields are typically non–divergent, and in many important examples the velocity field is approximately 2–D. The results presented here will also have relevance to problems like spatial interpolation of velocity data and stochastic simulation of non–divergent velocity fields.

This report is written for readers not familiar with the general theory of random vector fields, as presented in classical works of Kolmogorov (1941) Yaglom (1962), Yaglom (1987) or with the statistical theory of turbulence Batchelor (1953), Panchev (1971), Tatarskii (1971) and others. Some basic concepts for Gaussian random scalar fields are reviewed in Section 2. The appropriate relations for 2–D non–divergent random vector fields are developed in Section 3. In Section 5, the main results are summed up. Extension to random fields with homogeneous increments are discussed in Appendix A.1. In Appendix A.2, it is proved that a non–divergent homogeneous random vector field is uncorrelated with *any* homogeneous and isotropic scalar field.

2 Gaussian Random Scalar Fields

In this section concepts of *homogeneity* and *isotropy* for a random field will be introduced. Also, the mean square (m.s.) derivative of a random field will be defined and its relation to the covariance function will be explained.

Consider a spatial random field $\Psi(\boldsymbol{x})$, where $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j}$ is the location within some geographic region. The notation for the first two moments of the Ψ–field is:

$$\begin{aligned} \mathrm{E}\ \{\Psi(\boldsymbol{x})\} &= \mu(\boldsymbol{x}) && (1) \\ \mathrm{Cov}\ \{\Psi(\boldsymbol{x}'),\ \Psi(\boldsymbol{x}'')\} &= F_{\Psi}(\boldsymbol{x}', \boldsymbol{x}''). && (2) \end{aligned}$$

Analogous to the definition of "stationarity" for a stochastic process, the scalar field Ψ is *homogeneous* if it has a constant mean μ and the covariance function only depends on the separating vector $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}''$:

$$\begin{aligned} \mathrm{E}\ \{\Psi(\boldsymbol{x})\} &= \mu, \\ \mathrm{Cov}\ \{\Psi(\boldsymbol{x}'),\ \Psi(\boldsymbol{x}'')\} &= F_{\Psi}(\boldsymbol{r}). \end{aligned}$$

A homogeneous scalar field Ψ is *isotropic* if the covariance function only depends on the distance $r = \|\boldsymbol{r}\|$ between locations $\boldsymbol{x}'$ and $\boldsymbol{x}''$:

$$\mathrm{Cov}\ \{\Psi(\boldsymbol{x}'),\ \Psi(\boldsymbol{x}'')\} = F_{\Psi}(r).$$

The mean square (m.s.) partial derivative of $\Psi(\boldsymbol{x})$ in direction $\boldsymbol{i}$ is defined by the m.s. limit

$$\frac{\partial \Psi(\boldsymbol{x})}{\partial x} = \underset{h \to 0}{\text{l.i.m.}}\ \frac{\Psi(\boldsymbol{x} + h\boldsymbol{i}) - \Psi(\boldsymbol{x})}{h}, \tag{3}$$

or, equivalently, by the "ordinary" limit

$$\lim_{h \to 0} \mathrm{E}\ \{\ \frac{\partial \Psi(\boldsymbol{x})}{\partial x} - \frac{\Psi(\boldsymbol{x} + h\boldsymbol{i}) - \Psi(\boldsymbol{x})}{h}\ \}^2 = 0,$$

with a similar expression for the mean square (m.s.) partial derivative of $\Psi(\boldsymbol{x})$ in direction $\boldsymbol{j}$.

By using the definitions (2) and (3), the covariances between the m.s.–derivatives of $\Psi(\boldsymbol{x})$ can be expressed

$$\mathrm{Cov}\ \{\ \frac{\partial \Psi(\boldsymbol{x}')}{\partial x'}, \frac{\partial \Psi(\boldsymbol{x}'')}{\partial x''}\ \} = \frac{\partial^2 F_{\Psi}(\boldsymbol{x}', \boldsymbol{x}'')}{\partial x'\, \partial x''}$$

$$\begin{aligned}
\mathrm{Cov}\left\{ \frac{\partial\Psi(\boldsymbol{x}')}{\partial y'}, \frac{\partial\Psi(\boldsymbol{x}'')}{\partial y''} \right\} &= \frac{\partial^2 F_\Psi(\boldsymbol{x}', \boldsymbol{x}'')}{\partial y' \, \partial y''} \\
\mathrm{Cov}\left\{ \frac{\partial\Psi(\boldsymbol{x}')}{\partial x'}, \frac{\partial\Psi(\boldsymbol{x}'')}{\partial y'} \right\} &= \frac{\partial^2 F_\Psi(\boldsymbol{x}', \boldsymbol{x}'')}{\partial x' \, \partial y'}.
\end{aligned}$$

Thus, the covariance between the m.s. derivatives can be expressed by the ordinary derivatives of the covariance function F_Ψ. If $\Psi(\boldsymbol{x})$ is homogeneous these expressions can be simplified by using the chain rule for partial derivatives. The result is

$$\begin{aligned}
\mathrm{Cov}\left\{ \frac{\partial\Psi(\boldsymbol{x}')}{\partial x'}, \frac{\partial\Psi(\boldsymbol{x}'')}{\partial x''} \right\} &= -\frac{\partial^2 F_\Psi(\boldsymbol{r})}{\partial r_x^2} \\
\mathrm{Cov}\left\{ \frac{\partial\Psi(\boldsymbol{x}')}{\partial y'}, \frac{\partial\Psi(\boldsymbol{x}'')}{\partial y''} \right\} &= -\frac{\partial^2 F_\Psi(\boldsymbol{r})}{\partial r_y^2} \\
\mathrm{Cov}\left\{ \frac{\partial\Psi(\boldsymbol{x}')}{\partial x'}, \frac{\partial\Psi(\boldsymbol{x}'')}{\partial y'} \right\} &= -\frac{\partial^2 F_\Psi(\boldsymbol{r})}{\partial r_x \, \partial r_y},
\end{aligned} \tag{4}$$

where $\boldsymbol{r} = r_x\boldsymbol{i} + r_y\boldsymbol{j}$. For a homogeneous field it is often convenient to introduce polar coordinates

$$\begin{aligned}
r_x &= r \, \cos\theta \\
r_y &= r \, \sin\theta
\end{aligned}$$

By using the chain rule for partial derivatives the following relationships can be worked out:

$$\begin{aligned}
\frac{\partial F_\Psi}{\partial r} &= \frac{\partial F_\Psi}{\partial r_x}\cos\theta + \frac{\partial F_\Psi}{\partial r_y}\sin\theta \\
\frac{\partial F_\Psi}{\partial \theta} &= -\frac{\partial F_\Psi}{\partial r_x} r \, \sin\theta + \frac{\partial F_\Psi}{\partial r_y} r \, \cos\theta \\
\frac{\partial^2 F_\Psi}{\partial r^2} &= \frac{\partial^2 F_\Psi}{\partial r_x^2}\cos^2\theta + 2\,\frac{\partial^2 F_\Psi}{\partial r_x \partial r_y}\cos\theta \, \sin\theta + \frac{\partial^2 F_\Psi}{\partial r_y^2}\sin^2\theta \\
\frac{\partial^2 F_\Psi}{\partial \theta^2} &= \frac{\partial^2 F_\Psi}{\partial r_x^2} r^2 \, \sin^2\theta - 2\,\frac{\partial^2 F_\Psi}{\partial r_x \partial r_y} r^2 \, \cos\theta \, \sin\theta + \frac{\partial^2 F_\Psi}{\partial r_y^2} r^2 \, \cos^2\theta \\
&\quad -\frac{\partial F_\Psi}{\partial r_x} r \, \cos\theta - \frac{\partial F_\Psi}{\partial r_y} r \, \sin\theta \\
\frac{\partial^2 F_\Psi}{\partial r \partial \theta} &= -\left(\frac{\partial^2 F_\Psi}{\partial r_x^2} - \frac{\partial^2 F_\Psi}{\partial r_y^2}\right) r \, \cos\theta \, \sin\theta + \frac{\partial^2 F_\Psi}{\partial r_x \partial r_y} r \, (\cos^2\theta - \sin^2\theta) \\
&\quad -\frac{\partial F_\Psi}{\partial r_x}\sin\theta + \frac{\partial F_\Psi}{\partial r_y}\cos\theta
\end{aligned} \tag{5}$$

For a homogeneous and isotropic random field, the derivatives with respect to θ will be zero. The above results will be useful in the next section, where the second order moment for a vector field derived from a homogeneous and isotropic random field will be discussed. Analogous results for a situation where the homogeneity and isotropy assumptions are relaxed are given in Appendix A.1.

3 Non–divergent Random Vector Fields

In this section, we will show how to derive a non–divergent gaussian random vector field from a gaussian random scalar field.

Consider a homogeneous spatial random field $\Psi(\boldsymbol{x})$ as defined in Section 2, and define a spatial random vector field $\boldsymbol{V}(\boldsymbol{x}) = U(\boldsymbol{x})\ \boldsymbol{i} + V(\boldsymbol{x})\ \boldsymbol{j}$ with components

$$U(\boldsymbol{x}) = -\frac{\partial \Psi}{\partial y}, \quad V(\boldsymbol{x}) = \frac{\partial \Psi}{\partial x}.$$

Then it is easily verified that this vector field has zero divergence:

$$\nabla \cdot \boldsymbol{V} = 0.$$

Therefore $\Psi(\boldsymbol{x})$ can be thought of as a random *stream function* and $\boldsymbol{V}(\boldsymbol{x})$ as a random *velocity* field of an incompressible fluid (see Tritton (1977)). Denote the covariance functions for the vector components by $F_{UU}(\boldsymbol{r})$ and $F_{VV}(\boldsymbol{r})$, and the cross–covariance by $F_{UV}(\boldsymbol{r})$. The goal is to express these covariances in terms of the covariance function $F_{\Psi}(r)$ of the homogeneous and isotropic stream function $\Psi(\boldsymbol{x})$. For a homogeneous stream function the relations (4) give

$$\begin{aligned} F_{UU}(\boldsymbol{r}) &= -\frac{\partial^2 F_{\Psi}(\boldsymbol{r})}{\partial r_y^2} \\ F_{VV}(\boldsymbol{r}) &= -\frac{\partial^2 F_{\Psi}(\boldsymbol{r})}{\partial r_x^2} \\ F_{UV}(\boldsymbol{r}) &= \frac{\partial^2 F_{\Psi}(\boldsymbol{r})}{\partial r_x\, \partial r_y}. \end{aligned} \tag{6}$$

These relationships can be reexpressed in polar coordinates. Denote by V_r and V_θ the components of $\boldsymbol{V}$ parallel and perpendicular to $\boldsymbol{r}$, respectively. Then

$$\begin{aligned} V_r(\boldsymbol{x}) &= U(\boldsymbol{x})\ \cos\theta + V(\boldsymbol{x})\ \sin\theta \\ V_\theta(\boldsymbol{x}) &= -U(\boldsymbol{x})\ \sin\theta + V(\boldsymbol{x})\ \cos\theta. \end{aligned}$$

Denote the longitudinal, transversal and cross–covariances for these components by $F_{V_rV_r}$, $F_{V_\theta V_\theta}$, and $F_{V_rV_\theta}$, respectively. By using (5) these covariances can be expressed

$$\begin{aligned} F_{V_rV_r}(r,\theta) &= -\frac{1}{r^2}\frac{\partial^2 F_{\Psi}}{\partial \theta^2} - \frac{1}{r}\frac{\partial F_{\Psi}}{\partial r} \\ F_{V_\theta V_\theta}(r,\theta) &= -\frac{\partial^2 F_{\Psi}}{\partial r^2} \\ F_{V_rV_\theta}(r,\theta) &= \frac{1}{r}\frac{\partial^2 F_{\Psi}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial F_{\Psi}}{\partial \theta}. \end{aligned} \tag{7}$$

The relations (7) are valid for a homogeneous Ψ–field. So far, only the homogeneity condition has been used. For a homogeneous and isotropic random Ψ–field the covariance is a function of r only, so the above expressions simplify to

$$
\begin{aligned}
F_{V_r V_r}(r) &= -\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} \\
F_{V_\theta V_\theta}(r) &= -\frac{\mathrm{d}^2 F_\Psi}{\mathrm{d}r^2} \\
F_{V_r V_\theta}(r) &= 0,
\end{aligned}
\tag{8}
$$

where $\mathrm{d}/\mathrm{d}r$ denotes "ordinary" derivatives. Equations (8) are the general relations for the covariances of the longitudinal and transversal components of a 2–D non–divergent vector field. We see that the longitudinal and transversal components of this vector field are uncorrelated. Next, we would like to introduce isotropy in the expressions for the U and V components of the $\boldsymbol{V}$–field. Combining equations (8) and (5) gives

$$
\begin{aligned}
F_{V_r V_r}(r) &= -\frac{\partial^2 F_\Psi}{\partial r_x^2}\sin^2\theta + 2\,\frac{\partial^2 F_\Psi}{\partial r_x \partial r_y}\cos\theta\,\sin\theta - \frac{\partial^2 F_\Psi}{\partial r_y^2}\cos^2\theta \\
F_{V_\theta V_\theta}(r) &= -\frac{\partial^2 F_\Psi}{\partial r_x^2}\cos^2\theta - 2\,\frac{\partial^2 F_\Psi}{\partial r_x \partial r_y}\cos\theta\,\sin\theta - \frac{\partial^2 F_\Psi}{\partial r_y^2}\sin^2\theta \\
0 &= -\left(\frac{\partial^2 F_\Psi}{\partial r_x^2} - \frac{\partial^2 F_\Psi}{\partial r_y^2}\right)\cos\theta\,\sin\theta + \frac{\partial^2 F_\Psi}{\partial r_x \partial r_y}(\cos^2\theta - \sin^2\theta)
\end{aligned}
$$

We substitute F_{UU}, F_{VV} and F_{UV} from (6) and solve these three equations for F_{UU}, F_{VV} and F_{UV}. The result is

$$
\begin{aligned}
F_{UU}(r,\theta) &= F_{V_r V_r}(r)\,\cos^2\theta + F_{V_\theta V_\theta}(r)\,\sin^2\theta \\
F_{VV}(r,\theta) &= F_{V_r V_r}(r)\,\sin^2\theta + F_{V_\theta V_\theta}(r)\,\cos^2\theta \\
F_{UV}(r,\theta) &= (F_{V_r V_r}(r) - F_{V_\theta V_\theta}(r))\,\cos\theta\,\sin\theta
\end{aligned}
\tag{9}
$$

Unlike the covariance for a homogeneous and isotropic scalar field, the covariances for the $\boldsymbol{V}$–field will depend on the direction θ of the separating vector $\boldsymbol{r}$. The above formulae can be put into matrix form, denote the covariance matrix of the $\boldsymbol{V}$–field by $F_{\boldsymbol{V}}(r,\theta)$. Then the above results can be written

$$
\begin{aligned}
F_{\boldsymbol{V}}(r,\theta) &= (F_{V_r V_r}(r) - F_{V_\theta V_\theta}(r))\begin{pmatrix} \cos^2\theta & \cos\theta\,\sin\theta \\ \cos\theta\,\sin\theta & \sin^2\theta \end{pmatrix} \\
&+ F_{V_\theta V_\theta}(r)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

or more compactly

$$F_{\boldsymbol{V}}(r,\theta) = (F_{V_r V_r}(r) - F_{V_\theta V_\theta}(r)) \ \frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \ + \ F_{V_\theta V_\theta}(r) \ \mathcal{I}, \tag{10}$$

$\mathcal{I}$ being the identity matrix. Equation (10) is the general form of the covariance matrix of a homogeneous and isotropic random vector field in 2–D. An alternative derivation based on tensor calculus is given in Batchelor (1953).

From (8) it is seen that the longitudinal and transversal covariances $F_{V_r V_r}$ and $F_{V_\theta V_\theta}$ are related through the first order differential equation

$$F_{V_\theta V_\theta} = \frac{\mathrm{d}}{\mathrm{d}r} \ (r \ F_{V_r V_r}), \tag{11}$$

which is a 2–D version of the so–called Karman formula (see Panchev (1971) or Tatarskii (1971)). The isotropy condition (10) and the non-divergence condition (11) can be combined into

$$F_{\boldsymbol{V}}(r,\theta) = (F_{V_r V_r}(r) - \frac{\mathrm{d}}{\mathrm{d}r} \ (r \ F_{V_r V_r}))\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \ + \ \frac{\mathrm{d}}{\mathrm{d}r} \ (r \ F_{V_r V_r}) \ \mathcal{I}, \tag{12}$$

thereby expressing the covariance matrix of the $\boldsymbol{V}$–field in terms of the longitudinal covariance only.

Substituting the longitudinal and transversal covariances from (8) in (9) gives

$$\begin{aligned} F_{UU}(r,\theta) &= -\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} \ \cos^2\theta - \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2} \ \sin^2\theta \\ F_{VV}(r,\theta) &= -\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} \ \sin^2\theta - \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2} \ \cos^2\theta \\ F_{UV}(r,\theta) &= -(\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} - \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2}) \ \cos\theta \ \sin\theta \end{aligned} \tag{13}$$

This can be written on matrix form

$$F_{\boldsymbol{V}}(r,\theta) = -\left(\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} \ - \ \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2}\right) \ \frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \ - \ \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2} \ \mathcal{I} \tag{14}$$

Equation (12) or (14) can be taken as the definition of a (2–D) non–divergent homogeneous and isotropic random vector field. For $F_{\boldsymbol{V}}$ to exist (i.e. for $\boldsymbol{V}$ to be homogeneous and isotropic), it is enough that $\Psi(\boldsymbol{x})$ has homogeneous and isotropic increments, see Appendix A.1.

In conclusion, we have seen that a gaussian homogeneous and isotropic random vector field is specified by specifying its mean (a constant vector) and, either a differentiable covariance function for the longitudinal component, or a twice differentiable variogram function for the stream function.

4 Example

The above analysis provides an easy way of calculating the covariance matrix for a non–divergent 2–D vector field given a twice differentiable homogeneous and isotropic scalar covariance function. As an example, consider a homogeneous and isotropic stream function with "gaussian" type covariance function:

$$F_{\Psi}(r) \;=\; c_1\, e^{-3\left(\frac{r}{a}\right)^2} \;=\; c_1\, \rho(r),$$

for c_1 and a constants. The longitudinal and transversal covariance functions are easily computed

$$\begin{aligned}
F_{V_r V_r}(r) &= -\frac{1}{r}\frac{\mathrm{d}F_{\Psi}}{\mathrm{d}r} = c_2\, \rho(r) \\
F_{V_\theta V_\theta}(r) &= -\frac{\mathrm{d}^2 F_{\Psi}}{\mathrm{d}r^2} = c_2 \left(1 - 6\left(\frac{r}{a}\right)^2\right) \rho(r),
\end{aligned}$$

with $c_2 = 6/a^2\; c_1$. The covariance functions $F_{V_r V_r}$ and $F_{V_\theta V_\theta}$ are shown in Figure 1 for $c_2 = 1$, $a = 1$. It is seen that the transversal covariance is negative at certain distances. This can be seen qualitatively from the expression for $F_{V_\theta V_\theta}$ by the following simple argument: $F_{V_\theta V_\theta}$ is the negative curvature of F_{Ψ}. All covariance functions F_{Ψ} decaying smoothly to zero with increasing r will necessarily have positive curvature for some r, thereby giving negative $F_{V_\theta V_\theta}$ for this r. Physically, the qualitative properties of $F_{V_r V_r}$ and $F_{V_\theta V_\theta}$ are related to the tendency of non-divergent flow to form eddies.

Using (13), the covariances in a cartesian frame of reference are readily obtained:

$$\begin{aligned}
F_{UU} &= c_2 \left(1 - 6\left(\frac{r_y}{a}\right)^2\right) \rho(r), \\
F_{VV} &= c_2 \left(1 - 6\left(\frac{r_x}{a}\right)^2\right) \rho(r), \\
F_{UV} &= c_2\, 6\frac{r_x r_y}{a^2}\rho(r).
\end{aligned}$$

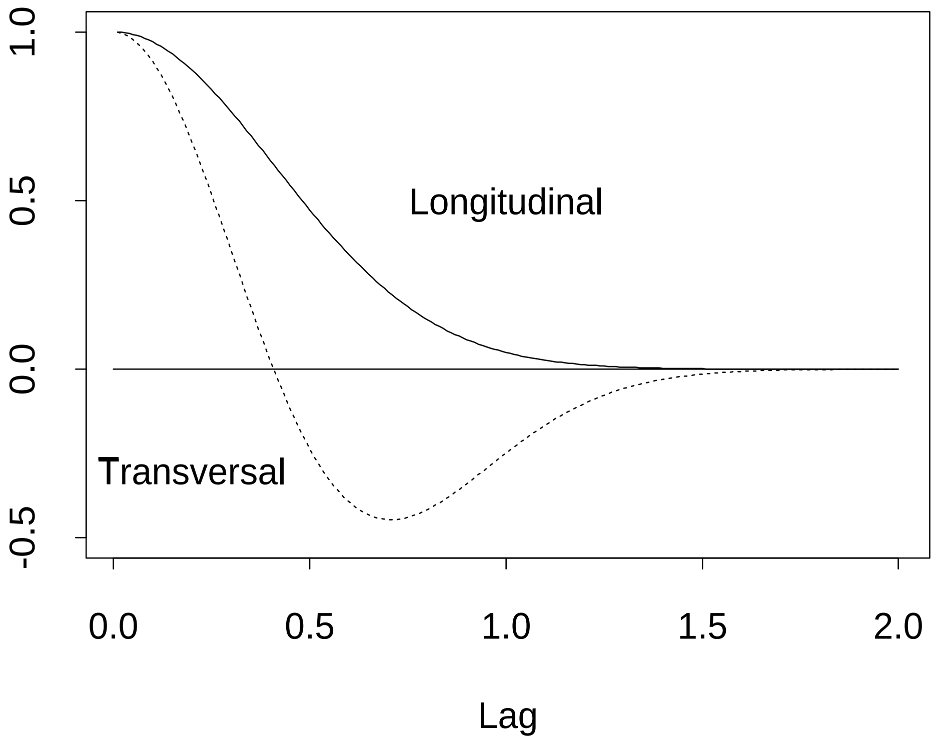

Figure 1: *Longitudinal and transversal components of the "gaussian" type covariance with parameters* $c_2 = 1$, $a = 1$.

5 Conclusions

We have derived expressions for the second moment of a 2–D non–divergent gaussian random vector field from an isotropic random stream function. It is seen that a homogeneous and isotropic covariance function for a random vector field can involve directions, unlike the scalar field case.

We have also shown that a 2–D non–divergent gaussian homogeneous random vector field can be specified by

- its first moment (a constant vector)
- its second moment, a 2×2 covariance matrix in terms of
  - either a twice differentiable homogeneous and isotropic variogram function

$$F_{\boldsymbol{V}}(r,\theta) = -\left(\frac{1}{r}\frac{\mathrm{d}F_\Psi}{\mathrm{d}r} - \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2}\right)\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' - \frac{\mathrm{d}^2F_\Psi}{\mathrm{d}r^2}\ \mathcal{I}$$

  - or a differentiable homogeneous and isotropic covariance function

$$F_{\boldsymbol{V}}(r,\theta) = (F_{V_rV_r}(r) - \frac{\mathrm{d}}{\mathrm{d}r}\ (r\ F_{V_rV_r}))\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' + \frac{\mathrm{d}}{\mathrm{d}r}\ (r\ F_{V_rV_r})\ \mathcal{I}.$$

In a similar way, a 2–D non–divergent gaussian non–homogeneous random vector field with homogeneous increments can be specified by (see the appendix)

- its first moment
- its second moment, a 2×2 variogram matrix expressed in terms of
  - either a twice differentiable homogeneous and isotropic generalized covariance function
  - or a differentiable homogeneous and isotropic variogram function.

A Appendix

A.1 Random fields with homogeneous and isotropic increments

For a spatial random field $\Psi(\boldsymbol{x})$ define the *variogram function* (also known as the *structure function*) D_Ψ

$$D_\Psi(\boldsymbol{x}', \boldsymbol{x}'') \;=\; \mathrm{Var}\;\{\Psi(\boldsymbol{x}') - \Psi(\boldsymbol{x}'')\}.$$

The random field has *homogeneous increments* if

$$D_\Psi(\boldsymbol{x}', \boldsymbol{x}'') \;=\; D_\Psi(\boldsymbol{r}),$$

for $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}''$. A random field $\Psi(\boldsymbol{x})$ has homogeneous and *isotropic* increments if

$$D_\Psi(\boldsymbol{r}) \;=\; D_\Psi(r)$$

A homogeneous and isotropic random field also has homogeneous and isotropic increments. For a homogeneous and isotropic random scalar field $\Psi(\boldsymbol{x})$, the covariance and variogram functions are related through

$$D_\Psi(r) \;=\; 2\,[F_\Psi(0) - F_\Psi(r)]$$

Calculations analogous to what has been shown for a homogeneous and isotropic stream function result in similar relationships for the vector field derived from a stream function Ψ with homogeneous and isotropic increments. Denote by $D_{\boldsymbol{V}}(\boldsymbol{r})$ the *variogram matrix* (structure matrix) of the vector field $\boldsymbol{V}(\boldsymbol{x})$ with homogeneous and isotropic increments. It is on the form

$$D_{\boldsymbol{V}}(r,\theta) = (D_{V_rV_r}(r) - D_{V_\theta V_\theta}(r))\;\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \;+\; D_{V_\theta V_\theta}(r)\;\mathcal{I},$$

where the longitudinal and transversal variogram functions are related through

$$D_{V_\theta V_\theta} = \frac{\mathrm{d}}{\mathrm{d}r}\,(r\;D_{V_rV_r})$$

In terms of $D_{V_rV_r}$ only:

$$D_{\boldsymbol{V}}(r,\theta) = (D_{V_rV_r}(r) - \frac{\mathrm{d}}{\mathrm{d}r}\,(r\;D_{V_rV_r}))\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \;+\; \frac{\mathrm{d}}{\mathrm{d}r}\,(r\;D_{V_rV_r})\;\mathcal{I}, \qquad (15)$$

or in terms of $D_\Psi$only

$$D_{\boldsymbol{V}}(r,\theta) = -\left(\frac{1}{r}\frac{\mathrm{d}D_\Psi}{\mathrm{d}r} \;-\; \frac{\mathrm{d}^2D_\Psi}{\mathrm{d}r^2}\right)\;\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}' \;-\; \frac{\mathrm{d}^2D_\Psi}{\mathrm{d}r^2}\;\mathcal{I} \qquad (16)$$

Equation (15) or (16) can be taken as the definition of a (2–D) non–divergent random vector field with homogeneous and isotropic increments. For $D_{\boldsymbol{V}}$ to exist (i.e. for $\boldsymbol{V}$ to have homogeneous and isotropic increments), it is sufficient that $\Psi(\boldsymbol{x})$ is an *intrinsic random field of order 1* (see Matheron (1973), Christakos (1992)). In this case, $D_{\boldsymbol{V}}$ can be specified by a twice differentiable generalized covariance function of order 1.

For a homogeneous and isotropic vector field $\boldsymbol{V}(\boldsymbol{x})$ the covariance and variogram matrices are related through

$$D_{\boldsymbol{V}}(\boldsymbol{r}) = 2\,[F_V(\boldsymbol{0}) \; - \; F_V(\boldsymbol{r})]$$

A.2 The covariance with a homogeneous, isotropic scalar field

Denote by $\boldsymbol{V}(\boldsymbol{x})$ a homogeneous, isotropic and non–divergent vector field and denote by $A(\boldsymbol{x})$ a homogeneous, isotropic scalar field. It will be shown that $\boldsymbol{V}(\boldsymbol{x})$ and $A(\boldsymbol{x})$ are uncorrelated.

By isotropy and the properties of a first order tensor (see Batchelor (1953)), the covariance between $\boldsymbol{V}(\boldsymbol{x})$ and $A(\boldsymbol{x})$ must be on the form

$$\mathrm{Cov}\,\{\boldsymbol{V}(\boldsymbol{x}'),\; A(\boldsymbol{x}'')\} \;=\; \frac{1}{r}\,\boldsymbol{r}\;C(r), \tag{17}$$

for $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}''$ and $r = \|\boldsymbol{r}\|$. The divergence of this covariance vector with respect to $\boldsymbol{x}'$ is

$$\begin{aligned} \nabla_{\boldsymbol{x}'} \cdot \mathrm{Cov}\,\{\boldsymbol{V}(\boldsymbol{x}'),\; A(\boldsymbol{x}'')\} &= \\ \mathrm{Cov}\,\{\frac{\partial U(\boldsymbol{x}')}{\partial x'},\; A(\boldsymbol{x}'')\} \;+\; \mathrm{Cov}\,\{\frac{\partial V(\boldsymbol{x}')}{\partial y'},\; A(\boldsymbol{x}'')\} &= \\ \mathrm{Cov}\,\{\nabla_{\boldsymbol{x}'} \cdot \boldsymbol{V}(\boldsymbol{x}'),\; A(\boldsymbol{x}'')\} &= \; 0, \end{aligned}$$

because $\boldsymbol{V}(\boldsymbol{x})$ is non–divergent.

On the other hand, the divergence of the right hand side of (17) is

$$\begin{aligned} \nabla_{\boldsymbol{x}'} \cdot \left[\frac{1}{r}\,\boldsymbol{r}\;C(r)\right] &= \\ C(r)\;\nabla_{\boldsymbol{x}'} \cdot \left(\frac{1}{r}\boldsymbol{r}\right) \;+\; \left(\frac{1}{r}\boldsymbol{r}\right) \cdot \nabla_{\boldsymbol{x}'} C(r) &= \\ \frac{1}{r}C(r) \;+\; \frac{\mathrm{d}C}{\mathrm{d}r} &= \; \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}[r\;C(r)] \end{aligned}$$

Combining the above results give

$$\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}[r\;C(r)] \;=\; 0.$$

This means that the covariance function $C(r)$ must be on the form

$$C(r) \;=\; \frac{\text{constant}}{r}$$

For $C(0)$ to be finite, this constant must be zero. This implies

$$\text{Cov}\;\{\boldsymbol{V}(\boldsymbol{x}'),\;A(\boldsymbol{x}'')\} \;=\; 0.$$

In particular, the stream function and its derived velocity field will be uncorrelated.

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