OPTIMAL REBALANCING OF PORTFOLIOS WITH TRANSACTION COSTS

HELGE HOLDEN AND LARS HOLDEN

Abstract. Rebalancing of portfolios with a concave utility function is considered. It is proved that transaction costs imply that there is a no-trade region where it is optimal not to trade. For proportional transaction costs it is optimal to rebalance to the boundary when outside the no-trade region. With flat transaction costs, the rebalance from outside the no-trade region should be to an internal state in the no-trade region but never a full rebalance. The standard optimal portfolio theory is extended to an arbitrary number of equally treated assets, general utility function, and more general stochastic processes. Examples are discussed.

1. Introduction

Financial portfolios usually consist of several different assets and rebalancing the portfolio is necessary in order to control the weights in the different assets. Most papers neglect transactions costs, but it is documented in, e.g., Atkinson *et al.* [1], Donohue and Yip [7] and Leland [9] that this may be a significant problem. In the present paper it is proved, assuming only concave utility function, that transaction costs imply that there is a no-trade (or no-transaction) region where it is not optimal to perform trading. If the transaction costs are proportional, then it is optimal to rebalance when outside the no-trade region to a state at the boundary of this region. If the transaction costs have fixed or flat elements, then the rebalance from outside the no-trade region should be to an internal state in the no-trade region but never a full rebalance. This extends previous results by considering an arbitrary number of equally treated assets, using a general utility function, fixed in addition to proportional transaction costs, and a larger class of stochastic processes. Most mathematical papers in the area use the viscosity solution of the Hamilton–Jacobi–Bellman equation to prove the existence and properties of the solution. Our approach is simpler. Existence is proved assuming the utility function is continuous and concave. The solution is found applying the Bellman principle at all times to maximize the utility. Furthermore, we provide an approximate formula and a more precise iterative simulation algorithm for the boundary of the no-trade region and the boundary for the internal states it should be rebalanced to in the case of fixed elements in the transaction costs.

The problem of optimally rebalancing a portfolio with transactions costs is studied in several papers, see Chang [4], Dybvig [5] and Liu [10] and references herein. The case of two assets can be analyzed analytically, see, e.g., Davis and Norman [6], Taksar *et al.* [18] and Øksendal and Sulem [16]. The multi-asset problem under strong assumptions has been studied by, e.g., Donohue and Yip [7]. Akian *et al.* [2] solve the multi-asset problem in an investment-consumption model with hyperbolic absolute risk aversion, Brownian motion and assuming proportional transaction costs using the viscosity solution of a variational inequality. Benth *et al.* [3] prove

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that the multi-asset problem with Lévy processes is well defined by using dynamic programming as well as the theory of viscosity solutions, but do not mention the no-trade region in the solution.

The more applied problem of finding the no-trade region in higher dimension is also discussed in several papers. Dynamic programming is prefered in most papers to identify the moving boundaries. Akian *et al.* [2], Muthuraman and Kumar [13] and Muthuraman and Zha [14] specify algorithms, but these do only work for very low dimension. More practical solutions are described in Sun *et al.* [17], Leland [9], and Fabozzi *et al.* [8]. Atkinson *et al.* [1] give an approximate formula for the boundary.

Most papers conclude that there is a no-trade region where trade should not be performed, and all rebalance from a state outside the no-trade region is to the boundary of the no-trade region. Donohue and Yip [7] formulate this as a general result for an arbitrary number of assets and only assuming concave utility function, but does not prove the result. Furthermore, they consider different rebalance strategies for portfolios with one risk-free asset and up to seven risky assets. Liu [10] gives a thorough discussion of the problem with constant absolute risk aversion and one risk-free investment and an arbitrary number of uncorrelated geometric Brownian motion investments. The paper discusses both proportional and fixed transaction costs and shows the existence of a no-trade region that is a fixed threshold for each investment with risk. All rebalance is between an investment with risk and the risk-free investment.

There are different formulations of the optimal portfolio problem. The classical paper Markowitz [11] employs a utility function defined on a portfolio. Leland [9] and Donohue and Yip [7] focus on portfolios with targets ratios and the utility function is deviance from the target ratios and the total transaction costs. Davis and Norman [6] and Liu [10] build on the work of Merton [12] that optimizes the consumption that is possible based on the portfolio. The properties of the different formulations appear quite similar, but with different mathematical challenges. All models seem to assume geometric Brownian motion. Section 2 describes our model, and the theorem regarding the no-trade region is proved. It extends the framework of Markowitz [11] with an arbitrary number of symmetric investments by also incorporating transaction costs in addition to a more general utility function and a larger class of stochastic processes. Different utility functions are discussed in Section 3. Our assumption with an arbitrary number of symmetric assets generalizes the assumption in Liu [10] where there is one risk-free asset at the cost of more complex analysis and complex boundary to the no-trade region. It includes the formulation of Leland [9] and the formulation of Merton [12] if we assume that consumption and investment are known.

The theoretical agreement regarding the existence of a no-trade region is in contrast to the current practice in most portfolios. Donohue and Yip [7] state that the typically reduction of transaction costs by using an optimal rebalance strategy is 50%. It seems to be most common to rebalance to what is considered the optimal balance at fixed time intervals, often monthly or each quarter, see, e.g., Leland [9]. Other portfolios define intervals for the weights in each asset and adjust to the boundary of these intervals either at fixed periods or continuously. Frequently, these decision criteria are often combined with a full rebalance at certain situations. In the Norwegian Government Pension Fund [15] the rebalance is mainly performed when deciding which new assets to buy. In addition, the portfolio is rebalanced to the target weights if the weights are outside certain intervals over two consecutive months. *Norges Bank* (The Norwegian Central Bank) states that as a large investor it is an advantage that the time of rebalances is not known in the marked and that the size of each rebalance is not too large. The strategy presented in this paper satisfies these criteria. The document from *Norges Bank* [15] includes a numerical simulation study of the transaction costs under different rebalancing strategies.

Sections 4 and 5 illustrate the theory with examples using different utility functions. The optimal weights and the no-trade region are described in both cases. In Section 4 analytic calculations are sufficient. In this section is it also described how to rebalance a portfolio to the boundary of the no-trade region. With the utility function discussed in Section 5 it is necessary with approximations and simulation. A procedure on how to determine the no-trade region approximately is provided, and it is shown how to improve this approximation by simulation. This example shows that the transaction costs are reduced to 1/4 by the use of an optimal no-trade region. Closing remarks are presented in Section 6.

2. The model

Consider a fixed but arbitrary number n of assets, and let $V_{i,t}$ denote the stochastic value of asset i, for $i = 1, 2, \ldots, n$ at time t. We assume that the stochastic properties of $V_{i,t}$ are known, and that the processes are Markovian, i.e., if $V_{i,t}$ is known, we do not get more information regarding the value at a later point in time by knowing the value of $V_{i,s}$ for all prior times, that is, for $s < t$. Since all assets are treated in the same way, we consider the problem as symmetric in the n assets. If $\log(V_{i,t})$ are correlated Lévy processes, all the utility functions defined in Section 3 are well-defined. Lévy processes include Brownian motion.

The portfolio is given by $\mathbf{a}_t = (a_{1,t}, a_{2,t}, \ldots, a_{n,t})$ where $a_{i,t}$ is the number of asset i at time t in the portfolio. The value of the portfolio at any given time t equals

$$W_t(\mathbf{a}_t) = \sum_{i=1}^{n} a_{i,t} V_{i,t} = \mathbf{a}_t \cdot \mathbf{V}_t. \tag{1}$$

Here we write $\mathbf{V}_t = (V_{1,t}, V_{2,t}, \ldots, V_{n,t})$. The weights are defined by

$$r_{i,t} = \frac{a_{i,t} V_{i,t}}{W_t}. \tag{2}$$

In order to simplify the formulation we let $a_{i,t} \in \mathbb{R}$, i.e., both positive and negative values are considered, where negative values indicate a short position in the asset. The results will be similar if we consider only non-negative values.

The problem to be discussed in this paper consists in deciding *when* it is preferable to rebalance the portfolio and to determine the *optimal* composition of the new portfolio.

A rebalance at time t_j implies that the portfolio is changed from

$$W_{t_j-} = \sum_{i=1}^{n} a_{i,t_{j-1}} V_{i,t_j} \tag{3}$$

to

$$W_{t_j} = \sum_{i=1}^{n} a_{i,t_j} V_{i,t_j} \tag{4}$$

where a_{i,t_j} denote the number of asset i in the time interval (t_j, t_{j+1}). In the case there are discontinuities in the value of the assets, $V_{i,t}$, we will always let $V_{i,t}$ denote the limit from the right. This also applies to (3). The reason for this definition is that if a jump in $V_{i,t}$ implies a rebalance, this is performed immediately, and obviously based on the values after the jump. Assume that

$$W_{t_j} = W_{t_j-} - c(D_{t_j}) \tag{5}$$

where the function $c(D_{t_j}) \geq 0$ is the cost of selling or buying assets at time t_j, which we denote by the rebalancing cost or transaction cost. In the case of a positive transaction cost c, transactions will happen in discrete times that we denote by t_j. With no transaction costs there will be a continuous rebalance, and we will not discuss that any further in this paper, except in Theorem 2.2 (B). The set D_t contains all relevant information or data regarding the assets up to time t, i.e.,

$$D_t = \{(\mathbf{a}_s, \mathbf{V}_s) \mid s \leq t\}.$$

We will assume that the transaction costs have proportional and fixed terms, i.e.,

$$c(D_{t_j}) = \sum_{i=1}^{n} \big(c_{i,1}|a_{i,t_j} - a_{i,t_{j-1}}|V_{i,t_j} + c_{i,2}\chi(a_{i,t_j} - a_{i,t_{j-1}})\big) \tag{6}$$

where $c_{i,j} \geq 0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2$. The function $\chi(a_{i,t_j} - a_{i,t_{j-1}}) = 1$ if $a_{i,t_j} \neq a_{i,t_{j-1}}$, i.e., if there is a rebalance in asset i at time t_j, and it equals 0 otherwise. For each asset i the cost function consists of a fixed fee if $c_{i,2} \neq 0$ and a cost proportional to the change in number of assets if $c_{i,1} \neq 0$. This formula covers the properties of interest from a theoretical point of view. It is trivial to extend it to the case when the transaction costs depend on whether we sell or buy an asset. To include taxation in the model may complicate the cost function if taxes depend on the time of acquisition of the asset and the difference between the value when sold and bought.

The quantity $\mathbf{a_s}(\mathbf{a}_t, z)$ for $s > t$ denotes the time development (at time s) of the portfolio which equals $\mathbf{a}_t$ at time t. The portfolio is a stochastic function and depends on the rebalance strategies z that are applied. In a similar way we write the value of the portfolio

$$W_s(\mathbf{a}_s(\mathbf{a}_t, z)) = \mathbf{a_s}(\mathbf{a}_t, z) \cdot \mathbf{V}_s.$$

There is a plethora of strategies. The simplest possible strategy is never to rebalance, that is, $a_{i,t}$ remains constant in time. Another natural strategy is to rebalance in order to have the weights within a given interval. We denote the said interval by $[\hat{\mathbf{r}}^1, \hat{\mathbf{r}}^2]$, with $\hat{\mathbf{r}}^j = (\hat{r}_1^j, \cdots, \hat{r}_n^j)$, in the sense that

$$\hat{r}_i^1 \leq r_{i,t_j} \leq \hat{r}_i^2,$$

or

$$\frac{\hat{r}_i^1}{V_{i,t_j}} W_{t_j} \leq a_{i,t_j} \leq \frac{\hat{r}_i^2}{V_{i,t_j}} W_{t_j}.$$

For this strategy it is necessary with detailed rules to determine how the rebalance is performed, e.g., only rebalance when at least one weight reaches the boundary of the admissible interval. Other strategies are of course also possible. We assume Markov strategies, i.e., the strategy z is a function only depending on the present situation and not the entire previous history; viz.

$$\mathbf{a}_{t_j} = S_z(\mathbf{a}_{t_{j-1}}, \mathbf{V}_{t_j}). \tag{7}$$

We assume the same strategy is applied at every point in time where it is possible to rebalance. Hence, a strategy is a complete description of when and how to rebalance, and we assume the same strategy is applied now and for every moment in the future. This may imply that it is rebalanced very often as is usual in hedging strategies. The set of admissible strategies is denoted by Z. We restrict ourselves to Markov strategies.

The utility function $U(\mathbf{a}_t, z, t)$ is assumed to be a real-valued deterministic function of the portfolio $\mathbf{a}_t$, the strategy z, and time t. We suppress the dependence on the time horizon T in the notation. It is natural that the utility increases in $E\{W_T(\mathbf{a}_T(\mathbf{a}_t, z))\}$ for some value of $T > t$ and decreases with the variability. See

Section 3 for a discussion on possible utility functions. At each rebalance time t the investor optimizes $U(\mathbf{a}_t, z, t)$. Many utility functions depend only on the value of the portfolio $W_s(\mathbf{a}_s(\mathbf{a}_t, z))$ for $s > t$. But if we include deviance from a reference portfolio in the utility, then also the number of assets, $\mathbf{a}_t$, is important in itself.

We want as general formulation of the utility as possible and need the following definitions:

Definition 2.1. *We collect the basic definitions below:*
We are given n assets each with value $V_{i,t}$, $i = 1, \dots, n$, at time t. All assets are treated in the same manner. The values of the assets are given by stochastic Markov processes with or without jumps.
(i) A portfolio is given by the collection of assets $\mathbf{a}_t = (a_{1,t}, a_{2,t}, \dots, a_{n,t})$. The value W_t of the portfolio is given by equation (1). The investor has the opportunity to rebalance the portfolio according to the equations (3)–(6) at arbitrary times $t_1, t_2, \dots$ by choosing the optimal points in time for rebalance and the optimal rebalance each time.

(ii) Let $z \in Z$ denote the investment strategy. We only consider Markov strategies $\mathbf{a}_{t_j} = S_z(\mathbf{a}_{t_{j-1}}, \mathbf{V}_{t_j})$, i.e., the state after the rebalance depends only on the state prior to the rebalance, namely $\mathbf{a}_{t_{j-1}}$, and the value of each asset, i.e., $\mathbf{V}_{t_j}$ at the time t_j of rebalance. We assume the same strategy is applied at each possible decision point.

(iii) A utility function is a real-valued deterministic function $U(\mathbf{a}_t, z, t)$ only depending on $\mathbf{a}_t$, the strategy z, and time t. We assume the stochastic processes $\mathbf{V}_t$ and the strategy z are sufficiently regular such that U is well-defined.

(iv) The utility $U(\mathbf{a}_t, z, t)$ is continuous/continuously differentiable/concave if $U(\mathbf{a}_t, z, t)$ is continuous/continuously differentiable/concave in $\mathbf{a}_t$ for every value of $z \in Z$. The utility function is concave in $\mathbf{a}_t$ if for any $\mathbf{a}^1$ and $\mathbf{a}^2$ satisfying $\mathbf{a}^1 \cdot \mathbf{V}_t = \mathbf{a}^2 \cdot \mathbf{V}_t = A$ where A is a constant, we have

$$U(\beta \mathbf{a}^1 + (1-\beta)\mathbf{a}^2, z, t) \geq \beta U(\mathbf{a}^1, z, t) + (1-\beta) U(\mathbf{a}^2, z, t) \tag{8}$$

for any constant $0 < \beta < 1$. We say that $U(\mathbf{a}_t, z, t)$ is strictly concave if (8) holds with strict inequality.

(v) The utility $U(\mathbf{a}_t, z, t)$ has compact level sets in the number of assets, $\mathbf{a}_t$, if for each combination of $A > 0$, $D \in \mathbb{R}$, and $z \in Z$, the set

$$\Omega_{A,D,z,t} = \{\mathbf{a} \in \mathbb{R}^n \mid \mathbf{a} \cdot \mathbf{V}_t = A, \quad U(\mathbf{a}, z, t) \geq D\}$$

is compact.

(vi) The transaction costs are super linear in $\mathbf{a}_t$ if the transaction costs due to a rebalance D_{t_j} from $\mathbf{a}_{t_{j-1}}$ to $\mathbf{a}_{t_j}$ satisfy

$$c(D_{t_j}) \geq B ||\mathbf{a}_{t_{j-1}} - \mathbf{a}_{t_j}||_{\mathbf{V}_t} \tag{9}$$

for a constant $B > 0$. We use the notation $||\mathbf{a}||_{\mathbf{V}_t} = \sum_{i=1}^n |a_i| V_{i,t}$.

(vii) The utility is super linear in $\mathbf{a}$ if there exists a constant $M > 0$ such that

$$U(\hat{\mathbf{a}}, z, t) \geq U(\mathbf{a}, z, t) + M ||\hat{\mathbf{a}} - \mathbf{a}||_{\mathbf{V}_t} \tag{10}$$

if $\hat{a}_i \geq a_i$ for all i and for all values of $z \in Z$.

(viii) The utility $U(\mathbf{a}, z, t)$ is homogeneous if $U(c\mathbf{a}, z, t) = cU(\mathbf{a}, z, t)$ for all constants c.

(ix) The utility $U(\mathbf{a}, z, t)$ is translation invariant in time[1] *for a strategy z if $U(\mathbf{a}, z, t) = cU(\mathbf{b}, z, t+\Delta t)$ when $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are such that $a_i V_{i,t}/W_t = b_i V_{i,t+\Delta t}/W_{t+\Delta t}$ for $i = 1, \dots, n$ for a constant c that only depends on t, $t+\Delta t$, W_t, and $W_{t+\Delta t}$.*

(x) For a strategy z and constant $A = W_t(\tilde{\mathbf{a}}_{A,z,t}) > 0$, we say that $\tilde{\mathbf{a}}_{A,z,t}$ is an optimal portfolio if it maximizes the utility, that is,

$$U(\tilde{\mathbf{a}}_{A,z,t}, z, t) = \sup_{\substack{\mathbf{a} \\ \mathbf{a}\cdot\mathbf{V}_t = A}} U(\mathbf{a}, z, t). \tag{11}$$

(xi) An optimal investment strategy $\tilde{z}$ is a strategy with the following property: It requires a rebalance at time t_j if and only if $U(\mathbf{a}_{t_{j-1}}, z, t_j)$ may be increased and then it is rebalanced to a value that optimizes $U(\mathbf{a}_{t_j}, z, t_j)$. More precisely,

$$U(\mathbf{a}_{t_j}, \tilde{z}, t_j) = \sup_{\hat{z}\in Z} U(S_{\hat{z}}(\mathbf{a}_{t_{j-1}}, \mathbf{V}_{t_j}), \hat{z}, t_j) \tag{12}$$

where $\mathbf{a}_{t_j} = S_{\tilde{z}}(\mathbf{a}_{t_{j-1}}, \mathbf{V}_{t_j})$.

Note that when there are transaction costs, equations (3)–(6) imply that $W_{t_j} < W_{t_j-}$.

(xi) A no-trade region is a region Ω_t where $\mathbf{a} \in \Omega_t$ if and only if it is impossible to increase $U(\mathbf{a}, z, t)$ by a rebalance including a change of strategy z and including the transaction costs at time t. More precisely

$$\begin{aligned} \Omega_t = \{\mathbf{a} \mid \mathbf{a} = S_{\tilde{z}}(\mathbf{a}, \mathbf{V}_t) \text{ and} \\ U(\mathbf{a}, \tilde{z}, t) = \sup_{\hat{z}\in Z} U(S_{\hat{z}}(\mathbf{a}, \mathbf{V}_t), \hat{z}, t)\}. \end{aligned} \tag{13}$$

The no-trade region may be defined both in terms of the number of assets, $\mathbf{a}$, and the weights, $\mathbf{r}$.

(xii) We say that the distribution has stationary relative increments if $V_{i,t+\Delta t}/V_{i,t}$ is independent of t for $i = 1, \dots, n$.

If the utility function has stationary relative increments, it may be more natural to use the weights $\mathbf{r}$ rather then the number of assets, $\mathbf{a}$, to describe the portfolio. If the portfolio has infinite time horizon, it is natural to choose a strategy z and a utility function U that is translation invariant in time. Section 3 illustrates some translation invariant utility functions.

We may then formulate the following theorem.

Theorem 2.2. *Let $U(\mathbf{a}_t, z, t)$ be a utility function defined on a portfolio $\mathbf{a}_t$ and a Markov strategy z, that is continuously differentiable, concave and has compact level sets in the weights. Let the transactions costs, $c(D_t)$, be of the form (3)–(6). Then the following statements hold:*

(A) There is an optimal portfolio $\tilde{\mathbf{a}}_{A,z,t}$ for each fixed value of $A = \sum_i a_{i,t} V_{i,t} > 0$ of the portfolio and each given z. The optimal portfolio is unique when the utility function is strictly concave.

(B) If there are no rebalancing costs, i.e., $c(D_t) = 0$ for all values of t in (5), then the optimal investment strategy $\tilde{z}$ is at any time to rebalance to the optimal portfolio $\tilde{\mathbf{a}}_{A,\tilde{z},t}$ given by (11) at every point t in time. Here $A = \mathbf{a}_t \cdot \mathbf{V}_t$.

(C) If both the transaction costs, (9), and the utility function, (10), are super linear in the number of assets $\mathbf{a}_t$, then there is a non-trivial no-trade region, Ω_t, and $\tilde{\mathbf{a}}_{A,\tilde{z},t} \in \Omega_t$. The no-trade region contains an open set containing $\tilde{\mathbf{a}}_{A,\tilde{z},t}$ for a given

[1] This essentially states that the utility function is only a function $\mathbf{r}_t$, and a change in time or the value of the portfolio W_t only results in a proportional change in the utility.

$A = \mathbf{a}_t \cdot \mathbf{V}_t$.

(D) Consider the case when the transaction costs have $c_{i,2} = 0$ *for* $i = 1, \ldots, n$. *Then there exists an optimal strategy* $\tilde{z}$ *with the following property: It rebalances whenever the portfolio is outside the no-trade region, and in that case it rebalances to the boundary of the no-trade region.*

(E) Consider the case when the transaction costs have $c_{i,2} = 0$ *for* $i = 1, \ldots, n$, *and* $V_{i,t}$ *for* $i = 1, 2, \ldots, n$ *have stationary relative increments and the utility function is homogeneous and translation invariant in time. Then the no-trade region is time independent in the weights* $\mathbf{r}_t$.

(F) Consider the case when the transaction costs have $c_{i,j} > 0$ *for* $i = 1, \ldots, n$ *and* $j = 1, 2$ *and the utility function is super linear in the number of assets* $\mathbf{a}_t$, (10). *From a state outside the no-trade region it is then optimal to rebalance to an internal state in the no-trade region.*

Proof. (A) Since U has compact level sets in the number of assets, $\mathbf{a}_t$, there exists for each strategy z a portfolio $\tilde{\mathbf{a}}_{A,z,t}$ where the optimum is obtained. The optimum is unique when the utility function is strictly concave.

(B) Since U is continuous on the compact set $\Omega_{A,D,z,t}$, there exist parameters $(\tilde{\mathbf{a}}_{A,\tilde{z},t}, \tilde{z})$ that optimize $U(\mathbf{a}, z, t)$. More precisely,

$$U(\tilde{\mathbf{a}}_{A,\tilde{z},t}, \tilde{z}, t) = \sup_{\substack{(\mathbf{a},z) \in \mathbb{R}^n \times Z \\ \mathbf{a} \cdot \mathbf{V}_t = A}} U(\mathbf{a}, z, t) \tag{14}$$

for each value of t. When there are no transactions costs, it is optimal for each point in time $t > 0$ to rebalance to the optimal number for each asset.

(C) Consider a rebalance from $\mathbf{a}_{t_{j-1}}$ to $\mathbf{a}_{t_j}$. Observe first that

$$\Omega_t \supseteq \{\tilde{\mathbf{a}}_{A,\tilde{z},t} \mid A \in \mathbb{R}\}$$

where $\tilde{\mathbf{a}}_{A,\tilde{z},t}$ is defined by (14). We want to show that if $\mathbf{a}_{t_{j-1}}$ is sufficiently close to $\tilde{\mathbf{a}}_{A,\tilde{z},t_j}$, then $\mathbf{a}_{t_{j-1}}$ is inside a no-trade region. Define a third state $\hat{\mathbf{a}}_{t_j}$ such that $\hat{a}_{i,t_j} = a_{i,t_j}$ if $a_{i,t_j} \geq a_{i,t_{j-1}}$ and $\hat{a}_{i,t_j} = a_{i,t_j} + \beta(a_{i,t_{j-1}} - a_{i,t_j})$ if $a_{i,t_j} < a_{i,t_{j-1}}$ where β is determined such that

$$\hat{\mathbf{a}}_{t_{j-1}} \cdot \mathbf{V}_{t_j} = \mathbf{a}_{t_{j-1}} \cdot \mathbf{V}_{t_j} = \mathbf{a}_{t_j} \cdot \mathbf{V}_{t_j} + c(D_{t_j}),$$

where D_{t_j} denotes the rebalance from $\mathbf{a}_{t_{j-1}}$ to $\mathbf{a}_{t_j}$. These two equations may be rewritten such that

$$\begin{aligned} \beta \sum_{\substack{i \\ a_{i,t_{j-1}} > a_{i,t_j}}} (a_{i,t_{j-1}} - a_{i,t_j}) V_{i,t_j} \\ = \sum_{\substack{i \\ a_{i,t_{j-1}} > a_{i,t_j}}} (a_{i,t_{j-1}} - a_{i,t_j}) V_{i,t_j} + \sum_{\substack{i \\ a_{i,t_{j-1}} < a_{i,t_j}}} (a_{i,t_{j-1}} - a_{i,t_j}) V_{i,t_j} = c(D_{t_j}). \end{aligned} \tag{15}$$

Since $c(D_{t_j}) > 0$, the second equality implies that β is positive, while the first one gives that $\beta \in (0, 1)$. Thus $\hat{a}_{i,t_j} \geq a_{i,t_j}$ and $\hat{a}_{i,t_j}$ is between a_{i,t_j} and $a_{i,t_{j-1}}$ including the endpoints for all i. Since $\beta < 1$ and the transaction costs are super linear, we have the following relations

$$||\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_j}||_{\mathbf{V}_{t_j}} = c(D_{t_j}) \geq B||\mathbf{a}_{t_{j-1}} - \mathbf{a}_{t_j}||_{\mathbf{V}_{t_j}}$$

for a constant $B > 0$.

We have that $\hat{\mathbf{a}}_{t_j} \cdot \mathbf{V}_{t_j} = \mathbf{a}_{t_{j-1}} \cdot \mathbf{V}_{t_j} = A$. Consider now

$$U(\tilde{\mathbf{a}}_{A,\tilde{z},t_j}, \tilde{z}, t_j) = \sup_{\substack{\mathbf{a} \\ \mathbf{a} \cdot \mathbf{V}_t = A}} U(\mathbf{a}, \tilde{z}, t_j). \tag{16}$$

At the maximum point we know from the theory of Lagrange multipliers that the gradient of the utility is parallel to the gradient of the constraint, i.e., the gradient of the value of the portfolio. In mathematical terms

$$\nabla_{\mathbf{a}} U(\tilde{\mathbf{a}}_{A,\tilde{z},t_j}, \tilde{z}, t_j) = \mu \mathbf{V}_{t_j},$$

for some constant μ. From the Taylor formula we see that (for some unspecified point ξ on the straight line between $\hat{\mathbf{a}}_{t_j}$ and $\mathbf{a}_{t_{j-1}}$)

$$\begin{aligned}
U(\hat{\mathbf{a}}_{t_j}, \tilde{z}, t_j) &= U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j) + \nabla_{\mathbf{a}} U(\xi, \tilde{z}, t_j) \cdot (\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_{j-1}}) \\
&= U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j) + \mu \mathbf{V}_{t_j} \cdot (\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_{j-1}}) + (\nabla_{\mathbf{a}} U(\xi, \tilde{z}, t_j) - \mu \mathbf{V}_{t_j}) \cdot (\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_{j-1}}) \\
&= U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j) + \nabla_{\mathbf{a}} \big(U(\xi, \tilde{z}, t_j) - U(\tilde{\mathbf{a}}_{A,\tilde{z},t_j}, \tilde{z}, t_j) \big) \cdot (\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_{j-1}}).
\end{aligned}$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||\nabla_{\mathbf{a}} \big(U(\tilde{\xi}, \tilde{z}, t_j) - U(\tilde{\mathbf{a}}_{A,\tilde{z},t_j}, \tilde{z}, t_j) \big)||_{\mathbf{V}_{t_j}} < \varepsilon$$

for all $\tilde{\xi}$ such that $||\tilde{\xi} - \tilde{\mathbf{a}}_{A,\tilde{z},t_j}||_{\mathbf{V}_{t_j}} < \delta$. If both $\hat{\mathbf{a}}_{t_j}$ and $\mathbf{a}_{t_{j-1}}$ satisfy

$$||\hat{\mathbf{a}}_{t_j} - \tilde{\mathbf{a}}_{A,\tilde{z},t_j}||_{\mathbf{V}_{t_j}} < \delta, \quad ||\hat{\mathbf{a}}_{t_{j-1}} - \tilde{\mathbf{a}}_{A,\tilde{z},t_j}||_{\mathbf{V}_{t_j}} < \delta$$

then also $||\xi - \tilde{\mathbf{a}}_{A,\tilde{z},t_j}||_{V_{t_j}} < \delta$, and we infer that

$$U(\hat{\mathbf{a}}_{t_j}, \tilde{z}, t_j) \le U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j) + \varepsilon ||\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_{j-1}}||_{\mathbf{V}_{t_j}}. \tag{17}$$

Using that the utility function increases at least linearly in the weights and that ε may be arbitrary small, we have

$$\begin{aligned}
U(\mathbf{a}_{\mathbf{t_j}}, \tilde{z}, t_j) &\le U(\hat{\mathbf{a}}_{t_j}, \tilde{z}, t_j) - M ||\hat{\mathbf{a}}_{t_j} - \mathbf{a}_{t_j}||_{\mathbf{V}_{t_j}} \\
&\le U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j) + \varepsilon ||\mathbf{a}_{t_{j-1}} - \hat{\mathbf{a}}_t||_{\mathbf{V}_{t_j}} - MB ||\mathbf{a}_{t_j} - \mathbf{a}_{t_{j-1}}||_{\mathbf{V}_{t_j}} \\
&\le U(\mathbf{a}_{t_{j-1}}, \tilde{z}, t_j).
\end{aligned}$$

Hence, it is not possible to increase the utility by rebalancing from $\mathbf{a}_{t_{j-1}}$ to $\mathbf{a}_{t_j}$. The above calculation is valid for any strategy. Since it is not possible to increase the utility by a rebalance from $\mathbf{a}_{t_{j-1}}$, there must be a no-trade region containing $\tilde{\mathbf{a}}_{A,\tilde{z},t_j}$. The same argument implies that all portfolios on the line between $\mathbf{a}_{t_{j-1}}$ and $\mathbf{a}_{t_j}$ with the same value of the portfolio also are inside the no-trade region. Hence, the no-trade region contains an open set containing $\tilde{\mathbf{a}}_{A,\tilde{z},t_j}$ for a given $A = \mathbf{a} \cdot \mathbf{V}_t$.

(D) We will prove that if the transaction costs only have proportional elements and no flat elements (i.e., $c_{i,1} > 0$ and $c_{i,2} = 0$), then all rebalances are to the boundary of the no-trade region. We will first show that there is an optimal rebalance strategy. The definition of no-trade region implies that the optimal strategy is to rebalance whenever outside the no-trade region. The rebalance will always be to a state inside or on the boundary of the no-trade region, since otherwise it will immediately rebalance to another state. The utility has compact level sets for a given value A of the portfolio. The no-trade region is a subset of such a level set. Furthermore, the no-trade region is a closed set. Thus the no-trade region is compact, and the utility function obtains its supremum inside the no-trade region. The transaction costs also vary continuously. Hence, the utility function, after a move into the no-trade region, including the transaction cost for the move, is a continuous function that obtains its maximum in the no-trade region. Hence, we have proved the existence of the optimal strategy. It is to rebalance whenever outside the no-trade region. In that case it rebalances to the supremum of the utility after a move, including the transaction costs of the move in the compact area of the no-trade region. The optimal strategy is denoted $\tilde{z}$.

Let $\mathbf{a}$ be outside the no-trade region and assume the optimal rebalance is to another state $\mathbf{d}$. By definition, the state $\mathbf{d}$ cannot also be outside the no-trade region, since then it will be possible to increase the utility by rebalancing from $\mathbf{d}$.

If $\mathbf{d}$ is in the interior of the no-trade region, we may define a third point $\mathbf{b}$ which is on the line between $\mathbf{a}$ and $\mathbf{d}$ and is on the boundary of the no-trade region. Let $c_{\mathbf{a},\mathbf{b}}$, be the transaction costs between $\mathbf{a}$ and $\mathbf{b}$. When the transaction costs are proportional, then $c_{\mathbf{a},\mathbf{d}} = c_{\mathbf{a},\mathbf{b}} + c_{\mathbf{b},\mathbf{d}}$. Then we may consider a rebalance from $\mathbf{a}$ to $\mathbf{d}$ to first be a rebalance from $\mathbf{a}$ to $\mathbf{b}$ and then from $\mathbf{b}$ to $\mathbf{d}$. From the assumptions we have $U(\mathbf{a}, \tilde{z}, t) < U(\mathbf{d}, \tilde{z}, t)$. Since $\mathbf{b}$ is on the boundary of the no-trade region and $\mathbf{d}$ is inside the region, we have $U(\mathbf{b}, \tilde{z}, t) \geq U(\mathbf{d}, \tilde{z}, t)$. This implies that we will not be able to improve the rebalance by rebalancing to the interior of the no-trade region compared to a rebalance to the boundary of the no-trade region.

(E) If all the elements that determine the no-trade region are time independent, then also the no-trade region is time independent. Let $r_{i,t_j-} = a_{t_{j-1}-}V_{i,t_j}/W_{t_j}$. If the transaction costs are proportional with the transaction and there are no flat elements (i.e., $c_{i,2} = 0$), then

$$c(D_{t_j}) = \sum_{i=1}^{n} |a_{i,t_j} - a_{i,t_{j-1}}| V_{i,t_j} = W_{t_j} \sum_{i=1}^{n} |r_{i,t_j} - r_{i,t_j-}|,$$

that is, the transaction costs are proportional with the value W_{t_j}.

If the utility is homogeneous, then the utility is proportional to W_t. If $V_{i,t}$ has stationary relative increments, then the development of the portfolio is independent of the value of $V_{i,t}$; it only depends on the weights $r_{i,t}$. Then it is natural to choose a strategy that only depends on the weights $\mathbf{r}_t$. If the utility is translation invariant in time, then the utility essentially only depends on $\mathbf{r}_t$. Under these assumptions, the no-trade region will only depend on the weights $\mathbf{r}_t$. If two portfolios have equal weights initially at $t = t_1$, that is, $\hat{r}_{i,t_1} = r_{i,t_1}$ for $i = 1, \ldots, n$, then they will have the same weights $\hat{r}_{i,t} = r_{i,t}$ for all $t > t_1$ if the same strategy z is applied.

(F) When all $c_{i,j} > 0$, the transaction costs are super linear. When we also assume that the utility is super linear, then the existence of the no-trade region follows from item (D). If $c_{i,2} = 0$ and the portfolio is outside the no-trade region, then it is optimal to rebalance to the boundary of the no-trade region. But then there may be a new rebalance very soon, if the portfolio moves outside the no-trade region once more. When $c_{i,2} > 0$, it cannot be optimal to perform many small rebalances. When outside the no-trade region, it is by definition optimal to perform a rebalance. In order to avoid a new rebalance soon, it is necessary to rebalance to an internal point in the no-trade region. $\square$

Notice that even a small transaction cost may change the strategy significantly. Without transaction costs, it may be optimal with continuous rebalancing, while with transaction costs, there is a no-trade region.

In a practical problem the investor will often increase or decrease the investment. This is an opportunity to rebalance to a lower additional cost than when there are no investment or consumption. Hence, these changes are important for the strategy to optimize the utility including reducing the transaction costs. If these changes are known in advance, the size of the no-trade region will increase when approaching the time when there is a change or the end point of the portfolio. The result may be that we only rebalance at these time-points.

In general, rebalancing implies that the investor sells assets that have increased in value over the last period. However, if the knowledge about the change in the value of an asset over a period implies a change in the expected further performance of the value of the asset, e.g., due to time dependent variance, it is critical that this is included in the model and the rebalancing strategy. It is well known that the optimal number of assets $\tilde{\mathbf{a}}_{A,z,t}$ is sensitive to small changes in the parameters in the stochastic process of the assets. If our expectation regarding the stochastic processes may change, e.g., due to new information, then this uncertainty should be included in the model. This will imply that the utility is more stable in $\mathbf{a}_t$, the no-trade region will be larger and the probability that we rebalance to weights that we soon after find out are far from optimal due to changes in the expected performance of the stochastic processes, is smaller.

Usually, there are some assets that have high expectations and high uncertainty. If these increase as expected, we may expect to sell these assets regularly in order to maintain the weights. However, if these assets have decreased in value, it may be optimal to wait and hope that these assets will increase in value in order that a rebalancing will not be necessary. This asymmetry is shown in Tables 1 and 4. See also discussion at the end of Section 5.1. The significance of this effect depends on the difference in expectation and the variability. With the utility function discussed in Section 4 however, the no-trade region is symmetric. This utility function only considers properties at the moment it is evaluated and not expected development later. Therefore, the difference in expectation between the different assets does not influence the no-trade region.

The size of the no-trade region depends on the importance of the transaction costs relative to the importance that the weights in the portfolio are close to the optimal number of assets, $\tilde{\mathbf{a}}_{A,z,t}$. By making the no-trade region larger, transactions will be less frequent and the associated costs go down. However, this also implies that the weights in the portfolio may be further from the optimal weights. The optimal size of the no-trade region may be difficult to assess, but it is not critical that we know the exact position since the derivative of the improvement vanishes at this point as we will see later in Figure 1. Assuming there is only proportional transaction costs, most of the reduction in transaction costs may be ensured if one rebalances only to the estimated boundary of the no-trade region, rather than a full rebalance to the optimal weights. In the case with fixed cost, there is obtained a similar improvement by rebalancing to an internal state in the no-trade region rather than a full rebalance to the optimal weights.

If any of the parameters changes, the optimal number of assets $\tilde{\mathbf{a}}_{A,z,t}$ changes and also the no-trade region changes position. If we neglect this change in position and consider an increase in volatility, then we may in general expect the no-trade region to increase in order not to rebalance too often and be more symmetric with respect to the optimal weights. See discussion at end of Section 5.1.

3. Utility functions

The definitions in this section are motivated by Markowitz [11]. An investor wants high expectation and low variability in the portfolio. Hence, the utility should increase in $E\{W_s\}$ for $s \geq t$ and decrease with the variability. Some authors argue that the utility function also should be decreasing in the expected deviance from a reference portfolio. If the portfolio ends at time T, it may be natural with the utility function

$$U(\mathbf{a}_t, z, t) = E\{W_T(\mathbf{a}_t, z)\} - d(\mathrm{var}(W_T(\mathbf{a}_{t_j}, z)))^{1/2} \tag{18}$$

or

$$U(\mathbf{a}_t, z, t) = E\{W_T(\mathbf{a}_t, z)\} - d\frac{\mathrm{var}(W_T(\mathbf{a}_t, z))}{W_t}. \tag{19}$$

The end date T is coming closer each day, implying that the no-trade region is gradually increasing, since the effect of more optimal number of assets decreases.

If we do not know when the portfolio is ended, it is natural to weigh the time with $\exp(-\beta t)$. Then the weights of the different time-points in the future are always the same and the no-trade region is stationary if the assets have stationary relative increments. This leads to the following utility function

$$U(\mathbf{a}_t, z, t) = \int_t^\infty (E\{W_s(\mathbf{a}_t, z)\} - d(\mathrm{var}(W_s(\mathbf{a}_t, z)))^{1/2}) \exp(-\beta s) ds \tag{20}$$

for constants $d > 0$ and $\beta > 0$. This utility function is proportional with a change in time. Other alternatives include value at risk $\mathrm{VaR}_\alpha W = P(W < \alpha)$, e.g.,

$$U(\mathbf{a}_t, z, t) = \int_t^\infty (E\{W_s(\mathbf{a}_t, z)\} - d\mathrm{VaR}_\alpha(W_s(\mathbf{a}_t, z))) \exp(-\beta s) ds, \tag{21}$$

or expected shortfall $E\{S_\alpha(W)\} = E\{W \mid W < \alpha\}$

$$U(\mathbf{a}_t, z, t) = \int_t^\infty (E\{W_s(\mathbf{a}_t, z)\} - dE\{S_\alpha(W_s(\mathbf{a}_t, z))\}) \exp(-\beta s) ds. \tag{22}$$

The utilities (20)–(22) are translation invariant for strategies z that only depend on the weights $\mathbf{r}$, rather than the assets $\mathbf{a}$, assuming the threshold α is suitably scaled. The threshold α may depend on the time s and the value of the portfolio $W_{s'}$ for $s' < s$, e.g.,

$$\alpha_s = 0.8 \max_{t \le s' \le \max\{t, s-1\}} \{W_{s'}\}.$$

It is possible to increase the investment $K(t) > 0$ or consume part of the investment $K(t) < 0$ by the following equation

$$W_t = W_{t-} + K(t) - c(D_t). \tag{23}$$

New investments and consumption are opportunities to rebalance with smaller additional transaction costs than to transfer between different assets. The properties of $K(T)$ for $T > t$ may be known at time t, may be stochastic or may be part of the optimization.

Merton [12] formulates the following utility function

$$U(\mathbf{a}_t, z, t) = E\{\int_t^T U_1(K_c, s) ds + U_2(W_T, T)\} \tag{24}$$

where the utility function U_1 is strictly concave in $-K_c$ (i.e., consumption) and U_2 is concave in W_T. The problem is both to rebalance the portfolio and to find the consumption $K_c \le 0$ that optimizes U. It is assumed to be one risk-free asset with no transaction costs. Liu [10] finds the constant thresholds of the no-trade region for each asset in this model, assuming uncorrelated geometric Brownian motion. All rebalance is a transfer between the risk-free asset and one of the other assets.

Leland [9] and Donohue and Yip [7] define ideal weights $\tilde{r}_i$ and use the utility function

$$U(\mathbf{a}_t, z, t) = E\{\sum_{i=1}^n b_i \int_t^T (r_{i,s}(\mathbf{a}_t, z) - \tilde{r}_i)^2 ds - \sum_j c(D_{t_j})\} \tag{25}$$

for constants $b_i > 0$ and where the last sum is over all times t_j where there is a rebalance. Instead of (5), it is assumed that transaction costs are paid by additional contributions. Leland [9] finds an approximation to the corner points in the no-trade region with this utility function assuming geometric Brownian motion, proportional transaction costs and one risk-free asset.

Note that the utility functions (18), (19), (20), (21), and (22) satisfy the assumptions in the theorem. The theorem is also valid for (25) and (24) if we fix the consumption. Properties (B)–(F) in the theorem depend on the transaction costs, and, in particular, whether the transaction costs (i) are zero, or (ii) are not identically zero; or (iii) do not contain fixed elements. If there are fixed elements in the transactions costs, it is optimal to rebalance to a point inside the no-trade region in order to avoid a new rebalance too soon. But the rebalance should never be to the optimal value $\tilde{\mathbf{a}}_{A,z,t}$ when the transaction costs have proportional elements in addition to the fixed elements, since the marginal improvement vanishes when the number of assets approaches $\tilde{\mathbf{a}}_{A,z,t}$.

4. An explicit example

In this section we illustrate the theorem by an example that is made so simple that it is possible to estimate the no-trade region mostly by analytic formulas. Let the utility function be

$$U(\mathbf{a}_t, z, t) = E\{(1 - \sum_{k=1}^{n} d_k(r_{k,t}(\mathbf{a}_t) - \tilde{r}_k)^2)W_t(\mathbf{a}_t)\}. \tag{26}$$

Here $\tilde{r}_k$ for $k = 1, \dots, n$ are given weights. With this utility function we always look for rebalances from $\mathbf{a}_{t-}$ to $\mathbf{a}_t$ such that the right-hand side of the expression above increases. The optimal values of the weights are obviously $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n)$, but in a rebalance we need to consider how much the value of the portfolio W_t is reduced compared to the improvement due to better weights.

We do not know the exact form of the no-trade region. It is natural to rebalance when the weight of one asset $r_{i,t}$ is high and the weight of another asset $r_{j,t}$ is low.

Consider a rebalance from asset i to asset j with $i \neq j$ at time t where the subscript $t-$ denotes the values just before the rebalance. Assume furthermore that before the rebalance we have $r_{i,t-} - r_{j,t-} > D_{i,j}$, and that we want to rebalance to

$$r_{i,t} - r_{j,t} = D_{i,j}. \tag{27}$$

Here $r_{i,t-} = a_{i,t-}V_{i,t}/\sum_j a_{j,t-}V_{j,t}$ while $r_{i,t}$ is defined in (2) after the rebalance. We will assume the new number of assets is $\mathbf{a_t} = \mathbf{a_{t-}} + \mathbf{b}$ where $b_i < 0$, $b_j > 0$, and $b_k = 0$, for $k \neq i, j$. With these restrictions on $\mathbf{b}$ equations (5) and (6) give

$$W_t = W_{t-} - c_{i,1}|b_i|V_{i,t} - c_{j,1}|b_j|V_{j,t} - c_{i,2} - c_{j,2}. \tag{28}$$

This equation has the solution, using equations (3) and (4),

$$b_i = \frac{-\eta - c_{i,2}}{(1 - c_{i,1})V_{i,t}} \tag{29}$$

and

$$b_j = \frac{\eta - c_{j,2}}{(1 + c_{j,1})V_{j,t}} \tag{30}$$

for a constant $\eta > c_{j,2}$. We find η by combining the above expression with (27). This gives the solution

$$\eta = \frac{a_{i,t-}V_{i,t} - a_{j,t-}V_{j,t} + \frac{c_{i,2}}{1-c_{i,1}} - \frac{c_{j,2}}{1+c_{j,1}} - D_{i,j}(W_{t-} + \frac{c_{i,2}}{1-c_{i,1}} + \frac{c_{j,2}}{1+c_{j,1}})}{D_{i,j}(\frac{1}{1-c_{i,1}} + \frac{1}{1+c_{j,1}}) - \frac{1}{1-c_{i,1}} + \frac{1}{1+c_{j,1}}}. \tag{31}$$

The above formula is useful when we want to determine the new weights in a rebalance.

If a rebalance involves more than two assets, it may be considered as several independent rebalances only involving two assets, when we assume there is only proportional transaction costs, i.e., $c_{i,2} = 0$ for all values of i. This makes it natural to assume that we may approximate the no-trade region by a region given by

$$H = \{\mathbf{r} \mid -D_{j,i} < r_i - r_j < D_{i,j} \text{ for all } i, j \text{ where } i \neq j, \text{ and } \sum_{k=1}^{n} r_k = 1\} \tag{32}$$

when there are only proportional transaction costs. We want to determine the constants $D_{i,j}$. Liu [10] proves in a similar model that the optimal no-trade region has corners with a similar geometry as H.

Equation (28) combined with the expression for the weights (2) gives the following expression for the utility function after a rebalance

$$U = (1 - \sum_{k=1}^{n} d_k (r_{k,t} - \tilde{r}_k)^2) \frac{W_{t-}(1 - r_{i,t-}c_{i,1} + r_{j,t-}c_{j,1}) - c_{i,2} - c_{j,2}}{1 - r_{i,t}c_{i,1} + r_{j,t}c_{j,1}}. \tag{33}$$

According to Theorem 2.2, it is optimal to rebalance whenever the weights are outside the no-trade region, and then it is rebalanced to the boundary of the no-trade region. We find $D_{i,j}$ for the values of $r_{i,t}$ and $r_{j,t}$ where

$$\frac{\partial U}{\partial r_{i,t}} - \frac{\partial U}{\partial r_{j,t}} = 0. \tag{34}$$

If we neglect higher-order terms in $c_{i,1}$ and $c_{j,1}$, we get, using (33), that

$$D_{i,j} = \frac{c_{i,1} + c_{j,1}}{d_i + d_j} + \tilde{r}_i - \tilde{r}_j. \tag{35}$$

Since we are able to find an expression for $D_{i,j}$ that is independent of $r_{k,t}$ for $k \neq i, j$, by neglecting higher order terms, this indicates that (32) is a good approximation to the no-trade region.

We will then consider the case with also a fixed transaction fee, i.e., $c_{k,2} > 0$. For simplicity we neglect the case where it is optimal to rebalance more than two assets at the same time. Assume the no-trade region may be approximated by a region on the form

$$G = \{\mathbf{r} \mid -E_{j,i} < r_i - r_j < E_{i,j} \text{ for all } i, j \text{ where } i \neq j \text{ and } \textstyle\sum_{i=1}^{n} r_i = 1 \ \}. \tag{36}$$

In this case it is optimal to rebalance whenever the weights are outside G, and then it should be rebalanced to the border of H since this is the optimal value when the flat fee is paid. This is according to Theorem 2.2, (F). We find $E_{i,j}$ from the values of $r_{i,t-}$ and $r_{j,t-}$ where the equation $U_- = U_+$ is satisfied. Here U_- is defined by (26) and U_+ is defined by (33) with $r_{i,t} - r_{j,t}$ defined by (27) and (34). This gives

$$E_{i,j} = 2\Big(\frac{c_{i,2} + c_{j,2}}{(d_i + d_j)W_{t-}} B + \frac{(c_{i,1} + c_{j,1})^2}{4(d_i + d_j)^2}(1 - B)^2\Big)^{1/2} + \frac{c_{i,1} + c_{j,1}}{d_i + d_j} B + \tilde{r}_i - \tilde{r}_j \tag{37}$$

where

$$B = \frac{4(d_i + d_j) - (c_{i,1} + c_{j,1})^2}{4(d_i + d_j)(1 - \tilde{r}_i c_{i,1} + \tilde{r}_j c_{j,1}) - 2(c_{i,1} + c_{j,1})^2} \approx 1.$$

This expression has the property that $E_{i,j} > D_{i,j}$ and

$$\lim_{c_{i,2} + c_{j,2} \to 0} E_{i,j} = D_{i,j}.$$

Note that $c_{i,2} + c_{j,2}$ scales with W_{t-}. Since the expression for $E_{i,j}$ is independent of r_k for $k \neq i, j$, this indicates that (36) is a good approximation for the no-trade region.

If we wanted to include rebalance of three assets at a time, we should define the set

$$G_3 = \{\mathbf{r} \mid -E^3_{i,k,j} < r_i - r_j - r_k < E^3_{i,j,k} \text{ for all different } i \text{ and } j > k \text{ and } \textstyle\sum_{i=1}^{n} r_i = 1\}. \tag{38}$$

We could define similar sets for rebalances involving $4, 5, \ldots$ assets at the same time. The no-trade region would then be defined as the intersection of G and the G_i sets. Then each set G_i contributes to a reduction of the no-trade region if there is a corresponding rebalance contribution to an increase in the utility. However, the

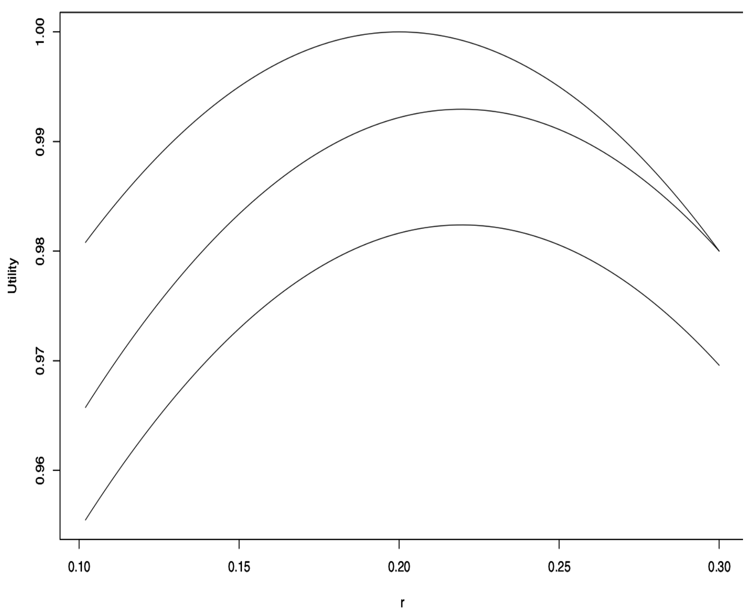

FIGURE 1. The top curve is the utility as a function of r and $W_t = 1$. This has optimum for $\tilde{r} = 0.2$. The two other curves show the utility that may be obtained from a rebalance from $r_{t-} = 0.3$. The top one of these has only proportional transaction fee while the lowest also has flat fee. Notice that both of these curves have optimum for the same value r that is the upper bound of the no-trade region with only proportional costs. The maximum of the lowest curves is equal to $U(0.3)$, illustrating that the boundary of the no-trade region including both proportional and flat transaction costs is 0.3. Since the two curves at the bottom are quite flat close to the optima, we see that the exact position of the no-trade region is not critical.

computational work increases significantly by including rebalances involving more than two assets.

This utility function is illustrated in Figure 1 with $n = 2$. The figure shows three utility curves, the utility as a function of $r_{1,t}$ and two utility curves that may be obtained with a rebalance from a $r_{1,t-} = 0.3$ assuming either only proportional transaction costs or both proportional and flat transaction costs. We have chosen constants $\tilde{r}_1 = 0.2$, $c_{k,1} = 0.04$ and $c_{k,2} = 0.0054$ for $k = 1, 2$. This implies the no-trade regions

$$H = \{r_1 \mid 0.18 < r_1 < 0.22\}$$

and

$$G = \{r_1 \mid 0.1 < r_1 < 0.3\}$$

when only proportional transaction fee and both proportional and flat transaction fee.

5. A numerical example

We will use the utility function

$$U(\mathbf{a}_t, z, t) = \int_t^\infty \big(E\{W_s(\mathbf{a}_s(\mathbf{a}_t, z))\} - d\frac{\mathrm{var}(W_s(\mathbf{a}_s(\mathbf{a}_t, z))}{W_t(\mathbf{a}_t)}\big)\exp(-\beta(s-t))ds. \tag{39}$$

The utility is homogeneous and proportional with a change in time and hence gives a stationary no-trade region Ω_t in the weights $\mathbf{r}_t$ if the assets have stationary relative increments. Many other utility functions will have similar properties. In this example, we have chosen to be more precise for this utility function instead of making the text more general. We assume the portfolio is evaluated once each day, and only then it is decided if one wants to rebalance.

We will first show how to find the optimal weights for this utility function. Then we find an approximation for the utility function that may be used in order to find an approximate no-trade region. We show that this approximation gives good estimates for the no-trade region. However, simulations may give better estimates.

Assume all assets are lognormally distributed and find optimal weights and an approximation for the no-trade region. Let $X_t = (X_{i,t}, X_{2,t}, \ldots, X_{n,t})$ where

$$X_t \sim N(\mu_X t, \Sigma_X t).$$

The correlation matrix is $\Sigma_X = \{\sigma^2_{X,i,j}\}_{i,j}$. Let

$$V_{i,t} = \exp(X_{i,t}),$$

and

$$\nu_i = \mu_{X,i} + \sigma^2_{X,i,i}/2.$$

Then we have the following

$$E\{V_{i,t}\} = \exp(\nu_i t),$$

and

$$\mathrm{Var}(V_{i,t}) = \exp(2\nu_i t)(\exp(\sigma^2_{i,i}t) - 1).$$

Recall that

$$W_t = \sum_i a_{i,t}V_{i,t}.$$

Assume we rebalance as often as possible such that

$$r_{i,t} = \frac{a_{i,t}V_{i,t}}{W_t}.$$

Define

$$\lambda = \sum_i r_{i,t}\nu_{X,i} \tag{40}$$

and

$$\gamma = 2\sum_i r_{i,t}\nu_{X,i} + \sum_{i,j} r_{i,t}r_{j,t}\sigma^2_{X,i,j}. \tag{41}$$

Then we have

$$E\{W_t\} = \exp(\lambda t), \tag{42}$$

and

$$E\{W_t^2\} = \exp(\gamma t). \tag{43}$$

Hence, the utility function (39) with no transaction costs and continuous rebalance to the optimal weights has the utility value

$$U = \frac{1}{\beta - \lambda} - d\Big(\frac{1}{\beta-\gamma} - \frac{1}{\beta - 2\lambda}\Big). \tag{44}$$

The optimal weights $\tilde{r}_{i,t}$ are the n variables that optimize (44), i.e., where $\partial U/\partial r_{i,t} = 0$ under the constraint that $\sum_i r_{i,t} = 1$. When there are no transaction costs it is optimal to rebalance continuously to the optimal weights $\tilde{r}_{i,t}$.

When there are transaction costs, there is a no-trade region according to Theorem 2.2. We assume there is only proportional transaction costs and that the no-trade region has the form (32). The optimal values of $D_{i,j}$ may only be found by CPU intensive simulation. However, we will find an approximation that does not need simulation. This approximation is described for $n = 2$ and later it is shown how to apply it for $n > 2$. In Section 4 it is described how to rebalance to the boundary of the no-trade region.

5.1. **Estimate U when $n = 2$.** Since the no-trade region is connected, the no-trade region for $n = 2$ is

$$H_2 = \{r_1 \mid D_- < r_1 < D_+\}, \tag{45}$$

only assuming that the utility function is concave and that the transaction costs has the form (6). In a simulation the ratio r_1 varies between the two limits D_- and D_+. If the ratio moves outside the interval, it is rebalanced to the boundary. We will first describe how r varies in this interval. Fix the time t and let $V_{1,\Delta t}$ and $W_{\Delta t}$ represent the change from t to $t + \Delta t$ in V_1 and W, respectively. We have

$$V_{1,t+\Delta t} = V_{1,t}V_{1,\Delta t} = V_{1,t}\exp(X_{1,\Delta t})$$

and

$$\begin{aligned} W_{t+\Delta t} = W_t W_{\Delta t} &= W_t(r_1 \exp(X_{1,\Delta t}) + (1 - r_1)\exp(X_{2,\Delta t})) \\ &\approx W_t(\exp(r_1 X_{1,\Delta t} + (1 - r_1)X_{2,\Delta t})). \end{aligned}$$

This gives

$$r_{1,t+\Delta t} = \frac{V_{1,t+\Delta t}}{W_{t+\Delta t}} = r_{1,t}\frac{V_{1,\Delta t}}{W_{\Delta t}} \approx r_{1,t}\exp((1 - r_{1,t})(X_{1,\Delta t} - X_{2,\Delta t})).$$

This implies with $\Delta r = r_{1,t} - r_{1,t-\Delta t}$ that

$$E\{\Delta r\} \approx r_{1,t-\Delta t}(E\{\exp((1 - r_{1,t-\Delta t})(X_{1,\Delta t} - X_{2,\Delta t}))\} - 1), \tag{46}$$

and

$$E\{(\Delta r)^2\} \approx r^2_{1,t-\Delta t}E\{(\exp((1 - r_{1,t-\Delta t})(X_{1,\Delta t} - X_{2,\Delta t})) - 1)^2\}. \tag{47}$$

We will approximate the distribution of Δr with a normal distribution $\phi(\Delta r) \sim N(\mu, \sigma^2)$. Let $p(r_1)$ be an approximation to the distribution of r_1 in the interval $[D_-, D_+]$. We want that $p(r_1)$ satisfies

$$p(r_1)\phi(\mu) = p(r_1 + \mu)\phi(-\mu), \quad r_1, r_1 + \mu \in (D_-, D_+). \tag{48}$$

The motivation for this is that density at r_1 multiplied by the probability to increase r_1 to $r_1 + \mu$ should be equal to the density at $r_1 + \mu$ multiplied by the probability to decrease from $r_1 + \mu$ to r_1 since $p(r_1)$ is independent of time. Equation (48) has the solution

$$p_1(r_1) = d_1 \exp(4\mu r_1/\sigma^2) \tag{49}$$

on the open interval (D_-, D_+). In addition, there is a positive probability that r_1 is equal the endpoints D_- and D_+. Therefore, we assume $p(r_1)$ has the form

$$p(r_1) = d_1 \exp(4\mu r_1/\sigma^2) + d_2\delta(D_-) + d_3\delta(D_+), \tag{50}$$

where

$$d_3 = d_3 P(\Delta r > 0) + P(r_{1,t} > D_+ \mid r_{1,t-\Delta t} < D_+)$$

$$= d_3 P(\Delta r > 0) + \int_{D_-}^{D_+} \int_{D_+ - r_1}^{\infty} p_1(r_1)\phi(\Delta r) d(\Delta r) dr_1.$$

We approximate $p_1(r_1)$ in the double integral with $p_1(D_+)$ and get

$$d_3 = \frac{p_1(D_+) \int_0^\infty \phi(\Delta r)\Delta r d(\Delta r)}{1 - P(\Delta r > 0)}.$$

Similarly, we find

$$d_2 = \frac{-p_1(D_-) \int_\infty^0 \phi(\Delta r)\Delta r d(\Delta r)}{1 - P(\Delta r < 0)}.$$

The three constants d_1, d_2 and d_3 are scaled such that $\int p(r_1)dr_1 = 1$.

Then the transaction costs due to the upper limit are

$$\int_{D_-}^{D_+} \int_{D_+ - r_1}^{\infty} p(r_1)\phi(r_{\Delta t})(r_1 + r_{\Delta t} - D_+)(c_{1,1} + c_{2,1}) d(\Delta r) dr_1$$
$$= (c_{1,1} + c_{2,1}) \int_0^\infty \phi(\Delta r) \int_{D_+ - \Delta r}^{D_+} p(r_1)(r_1 + \Delta r - D_+) dr_1 d(\Delta r)$$
$$\approx \frac{1}{2}(c_{1,1} + c_{2,1}) p(D_+) \int_0^\infty \phi(\Delta r)(\Delta r)^2 d(\Delta r).$$

Using a similar calculation for the transaction costs due to the lower limit, gives the following expression for the transaction costs in a time step

$$C_{\Delta t} = \frac{1}{2}(c_{1,1} + c_{2,1})(p(D_-) + p(D_+)) E\{r_{\Delta t}^2\}.$$

The expectation and variance of W_t are calculated as follows, assuming it follows the properties of lognormal distributions,

(51)
$$E\{W_t^q\} = (E\{W_{\Delta t}^q\})^{t/\Delta t} = (E\{W_{C,\Delta t}^q (1 - C_{\Delta t})^q\})^{N_d t} = (1 - C_{\Delta t})^{q N_d t} (E\{W_{C,\Delta t}^q\})^{N_d t}$$

where $q = 1, 2$. In this calculation, we have assumed the same relative increase in W_t in each time step. The variable $W_{C,\Delta t}$ is the increase in W in one time step Δt when we neglect the reduction due to transaction costs in this time step. Let $N_d = 1/\Delta t$ be number of time steps in a year. Let furthermore

$$\hat{\lambda}_2 = N_d \log(\sum_i p_i (R_i E\{V_{1,\Delta t}\} + (1 - R_i) E\{V_{2,\Delta t}\})) + N_d \log(1 - C_{\Delta t})$$

and

$$\hat{\gamma}_2 = N_d \log(\sum_i p_i E\{(R_i V_{1,\Delta t} + (1 - R_i) V_{2,\Delta t})^2)\} + 2 N_d \log(1 - C_{\Delta t})$$

where R_i and p_i for $i = 1, 2, \ldots, m$ is a discretization of $p(r)$ for $r \in (D_-, D_+)$. R_i denotes values in the interval and p_i the corresponding probability.

The expression $\exp(\hat{\lambda}_2 t)$ may be used as an estimate for $E\{W_t\}$ and the expression $\exp(\hat{\gamma}_2 t)$ as an estimate for $E\{W_t^2\}$. Hence, an estimate for the value of the utility function U when we rebalance outside the no-trade region defined in (45) is

$$\hat{U}_2 = \frac{1}{\beta - \hat{\lambda}_2} - d\left(\frac{1}{\beta - \hat{\gamma}_2} - \frac{1}{\beta - 2\hat{\lambda}_2}\right). \quad (52)$$

Instead of finding the two parameters D_-, D_+, it is more stable to find the parameters for the length of the no-trade region $D_+ - D_-$ and for the position relative to the optimal weight $\tilde{r}$, e.g., $P = \frac{\tilde{r} - D_-}{D_+ - D_-}$.

Assume asset 1 has higher expected increase than asset 2. Then $\mu > 0$ and the function p defined in (50) is increasing. Since the object function is quite symmetric around the optimal weights, it is more important that the right end point of the no-trade region is closer to the optimal weights than the left end point. Hence, the relative position will satisfy $P > 0.5$. This implies that when asset 1 has lower weight than the optimal, we are more reluctant to impose a rebalance than for asset 2, since asset 1 is more likely to increase without a rebalance. Notice also that if σ in (49) increases, then the slope of p decreases. Then P approaches 0.5, making the no-trade interval more symmetric.

Optimization of the above approximation does not give the optimal no-trade region for the utility function U. However, experiments show that the above approximation gives almost as large values of the utility function as when applying the optimal no-trade region. The approximation may also be used as a first guess on the no-trade region. Then it is possible to adjust these values based on a simulation if wanted.

5.2. **Estimate U when $n > 2$.** We approximate the no-trade region by a region on the form (32). For $n > 2$ we use the same technique for each pair r_i and r_j. The quantity $r_i - r_j$ varies between two boundaries $-D_{j,i} < r_i - r_j < D_{i,j}$. If we find an approximation to the pair of boundaries $-D_{j,i}, D_{i,j}$ separately from the other $D_{k,m}$, we may find $-D_{j,i}, D_{i,j}$ similarly as we found D_-, D_+. The optimal values may only be found by a simulation of all the boundaries simultaneously.

Let us fix i and j and set $\Delta r = r_{i,t} - r_{j,t} - r_{i,t-\Delta t} + r_{j,t-\Delta t}$. Similar to the argument for $n = 2$, we have for $n > 2$

$$\Delta r = r_{i,t-\Delta t}\big(\frac{V_{i,\Delta t}}{W_{\Delta t}} - 1\big) - r_{j,t-\Delta t}\big(\frac{V_{j,\Delta t}}{W_{\Delta t}} - 1\big).$$

This gives

$$E\{(\Delta r)^q\} = E\{(r_{i,t-\Delta t}(\frac{V_{i,\Delta t}}{W_{\Delta t}} - 1) - r_{j,t-\Delta t}(\frac{V_{j,\Delta t}}{W_{\Delta t}} - 1))^q\}. \tag{53}$$

We may use the approximation

$$\frac{V_{i,\Delta t}}{W_{\Delta t}} \approx \exp(X_{i,\Delta t} - \sum_{k=1}^{n} r_{k,t} X_{k,\Delta t}) \tag{54}$$

in order to find expressions for $E\{(\Delta r)^q\}$.

For $n = 2$, we defined discrete values R_i in the interval (D_-, D_+). Similarly, we define

$$R_{i,k} = \frac{1}{2}(-D_{j,i} - \tilde{r}_i + \tilde{r}_j) + \frac{1}{2m}(k - \frac{1}{2})(D_{i,j} + D_{j,i})$$

and $R_{j,k} = -R_{i,k} + \tilde{r}_i - \tilde{r}_j$ for $k = 1, 2, \ldots, m$. Then $R_{i,k} - R_{j,k}$ varies in the interval $(-D_{j,i}, D_{i,j})$. In order to simplify some expressions below we define $R_{q,k} = \tilde{r}_q$ for $q \neq i, j$ and for $k = 1, 2, \ldots, m$. This is a first order approximation to the average value of r_q. Let p_k be the probability for the discrete value $R_{i,k}$ defined using (53) and an expression similar (50) for $n > 2$. Note that we find p_k for each pair i, j. We then have the following approximations to λ and γ using (51)

$$\hat{\lambda} = N_d \log(\sum_{k=1}^{m} p_k \sum_{v=1}^{n} R_{v,k} E\{V_{m,\Delta t}\}) + N_d \log(1 - C_{\Delta t})$$

and

$$\hat{\gamma} = N_d \log(\sum_{k=1}^{m} p_k \sum_{v=1}^{n} \sum_{u=1}^{n} R_{v,k} R_{u,k} E\{V_{v,\Delta t} V_{u,\Delta t}\}) + 2N_d \log(1 - C_{\Delta t}).$$

We have

$$E\{V_{v,\Delta t}\} = \exp(\mu_{\Delta X,v} + \sigma^2_{\Delta X,v}/2)$$

and

$$E\{V_{v,\Delta} V_{u,\Delta t}\} = \exp(\mu_{\Delta X,v} + \mu_{\Delta X,u} + (\sigma^2_{\Delta X,v} + \sigma^2_{\Delta X,u})/2 + \sigma^2_{\Delta X,v,u}),$$

d	$c_{i,1}$	ρ	$\tilde{r}$	D_-	D_+
2.72	0.01	0	0.2	0.165	0.212
2.72	0.001	0	0.2	0.186	0.203
2.15	0.01	0.3	0.2	0.167	0.212

TABLE 1. The critical values for $n = 2$ with proportional transaction. The two assets are geometric Brownian motion with annual expectation and standard deviation equal to $\mu_V = (1.08, 1.02)$ and $\sigma_V = \text{diag}(0.2, 0.04)$, and discounted by $\beta = -\log(0.8)$.

where $\mu_{\Delta X,v}$, $\sigma^2_{\Delta X,v}$ and $\sigma^2_{\Delta X,v,u}$ correspond to the expectation and variance for variable v and the correlation between v and u per time step.

Hence, an approximation $\hat{U}$ to the value of U when we rebalance outside the no-trade region (32) is

$$\hat{U} = \frac{1}{\beta - \hat{\lambda}} - d\Big(\frac{1}{\beta - \hat{\gamma}} - \frac{1}{\beta - 2\hat{\lambda}}\Big). \tag{55}$$

This approximation may by used in order to find an approximation to the optimal boundaries $(-D_{j,i}, D_{i,j})$. In this approach we find the boundaries for each pair i, j separately. Also in this case, it is stable to parameterize the length of the interval $D_{j,i} + D_{i,j}$ and the position relative to the difference between the optimal weights $\tilde{r}_i - \tilde{r}_j$, e.g., $\frac{\tilde{r}_i - \tilde{r}_j + D_{j,i}}{D_{i,j} + D_{j,i}}$. Application of the above approximation will not give the optimal no-trade region for the object function U. But experiments show that the approximation gives almost the optimal value. The approximation may also be used in order to find a first guess on the no-trade region that is improved using simulation.

5.3. **Simulation of portfolios.** In this section we find the no-trade region by simulation using the utility function (39). The value of the assets $V_{i,t}$ is modeled by logarithmic Brownian motion. In the simulation it is rebalanced to the boundary of the no-trade region when the portfolio is outside the region using equation (31).

Table 1 shows the no-trade region for $n = 2$ for different values of the proportional transaction costs and with and without correlation between the different assets. Note that when adding correlation, it is necessary to change d in order to have optimum for $\tilde{r} = 0.2$. Table 2 compares the utility of four different strategies. It is shown that optimal rebalance gives highest utility.

Referring to Table 2, we see that the simulated optimal rebalance has as expected highest utility function, but using the approximation to the no-trade region gives only slightly lower utility. The optimal no-trade region, assuming the form (32), is only small adjustments of the no-trade region given by approximate formulas and is computed by simulation. The case of no rebalance increases the ratio of the high volatile asset giving higher expected value of portfolio $E\{W_1\}$ but at a cost of higher variance. The difference in variance increases faster than the difference in expected value. This is seen by comparing no rebalance and optimal rebalance after 1 year when we use the same d value as in the utility function. Optimal rebalance reduces the transaction costs to 1/4 compared to monthly rebalance.

The second example with $n = 5$ correlated assets and proportional transaction costs is shown in Tables 3–5. Table 3 shows the parameters for the five assets with values that are assumed realistic for the Norwegian stock marked, international stock marked, Norwegian real estate, Norwegian bonds, and international bonds. The table shows the optimal weights for the five assets with the utility function (39) and three different values of d. Table 4 shows the no-trade region for $d = 2$.

strategy	U	EW_1	U_1	C_n	C_c
no rebalance	4.8089	1.0325	1.0254	0	0
monthly rebalance	4.9230	1.0307	1.0240	12	0.0019
optimal rebalance (approximate)	4.9457	1.0318	1.0251	12	0.00039
optimal rebalance (simulated)	4.9460	1.0316	1.0254	13	0.00047

TABLE 2. Comparison of four different rebalancing strategies with $n = 2$. The parameters are as in the top row of Table 1 including the threshold for the optimal rebalance. The quantity U is an estimate for the utility function, $E\{W_1\}$ is the expected value and $U_1 = E\{W_1\} - d\text{var}(W_1)$ is another evaluation of the portfolio after 1 year, C_n and C_c denote the annual number of transactions and annual transaction costs, respectively.

i	μ_i	σ_i	$\rho_{i,1}$	$\rho_{i,2}$	$\rho_{i,3}$	$\rho_{i,4}$	$\rho_{i,5}$	$\tilde{r}_{i,d=0.5}$	$\tilde{r}_{i,d=1}$	$\tilde{r}_{i,d=2}$
1	1.1	0.22	1	0.7	0.1	0.3	0.1	0.223	0.152	0.083
2	1.09	0.20	0.7	1	0.05	0.1	0.2	0.175	0.129	0.092
3	1.05	0.12	0.1	0.05	1	0	0	0.255	0.201	0.157
4	1.035	0.04	0.3	0.1	0	1	0.3	0.071	0.193	0.306
5	1.035	0.04	0.1	0.2	0	0.3	1	0.0277	0.325	0.362

TABLE 3. The critical values for $n = 5$ with proportional transaction costs $c_{i,1} = 0.01$ and three different d values. The parameters are expectation, standard deviation, correlation, and the optimal values for each asset for the different values of d.

i/j	1	2	3	4	5
1	—	0.094	0.094	0.044	0.041
2	0.69	—	0.069	0.053	0.058
3	0.52	0.59	—	0.12	0.10
4	0.75	0.73	0.67	—	0.14
5	0.50	0.73	0.51	0.67	—

TABLE 4. The optimal no-trade region for the example shown in Table 3 with $d = 2$. Above the diagonal is the length L of the interval for $r_i - r_j$ and below the diagonal is the position P of the interval. The no-trade region is then $\tilde{r}_i - \tilde{r}_j - LP < r_i - r_j < \tilde{r}_i - \tilde{r}_j + L(1 - P)$.

strategy	U	$E\{W_1\}$	U_1	C_n	C_c
no rebalance	5.2745	1.04761	1.04228	0	0
monthly rebalance	5.3692	1.04549	1.04092	12	0.0022
optimal rebalance (approximate)	5.3988	1.04693	1.04239	36	0.00049
optimal rebalance (simulated)	5.4000	1.04696	1.04240	36	0.00050

TABLE 5. The table is exactly as Table 2 except that $n = 5$ and we have used data as in Table 3. It compares the strategies, no rebalance, monthly rebalance, and optimal rebalance with no-trade region found by approximation and by simulation. The optimal no-trade region is as in Table 4.

Note how the length L varies between the different combinations of assets and the position in some cases is far from the symmetric 0.5 value. Table 5 is similar to Table 2 but for the example with five assets. The table compares four different strategies and shows that the optimal rebalance strategy gives highest utility. The approximation to the no-trade region gives almost as good results as the optimal rebalance. The no-trade region is slightly different in the two cases and the result is not sensitive to the exact position. Note that also here there is a reduction of transaction costs to 1/4 compared to monthly rebalance. But in this case the expected annual number of rebalances is 36 which is much larger than in monthly rebalance.

The calculation in both examples is based on 10000 simulations until 10 years and then estimated tail for $t > 10$. For $n = 5$ one such simulation takes about 3 hours using the statistical package R on a standard desk top computer. Finding the optimal no-trade region by estimating the 20 parameters from a good starting point, requires at least 100 simulations which gives about 2 weeks of simulation time. The approach based on approximation took about 20 seconds which is an improvement compared with simulation of the order 10^5.

6. Closing remarks

This paper discusses optimal rebalance of portfolios with transaction costs. We have shown that for n symmetric assets and a general utility function, there is a no-trade region. n symmetric assets means that all assets are treated similarly with no particular assumption on any of the assets. If the transaction costs are proportional, with no flat or fixed elements, it is optimal to rebalance to the boundary of the no-trade region whenever the portfolio is outside the no-trade region. If the transactions costs have flat elements, it is optimal to rebalance to an internal surface in the no-trade region whenever the portfolio is outside the no-trade region. It is never optimal to have a full rebalance or a calendar-based rebalance.

The theory is illustrated by two examples; one using analytic calculations and approximations and one using simulations. The last example is simulated for $n = 2$ and $n = 5$. Three different rebalance strategies, namely, no rebalance, monthly rebalance, and optimal rebalance are tested using simulations. Both for $n = 2$ and $n = 5$ the transaction costs are reduced by a factor 4 compared to monthly rebalance. These figures are slightly better than other papers on optimal rebalance for a particular utility function. The reduction in transaction costs is probably mainly due to the fact that we rebalance to the boundary of the no-trade region instead of a full rebalance. It is probably not very critical to the exact position of the boundary of the no-trade region. But in order to optimize the utility, it is critical to have an optimal no-trade region. The example shows that the size of the no-trade region depends heavily on the properties of the stochastic processes, not only the size of the transaction costs.

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(H. Holden)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway, and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway
E-mail address: holden@math.ntnu.no
URL: http://www.math.ntnu.no/~holden/

(L. Holden)
Norwegian Computing Center, P. O. Box 114 Blindern, NO–0314 Oslo, Norway
E-mail address: lars.holden@nr.no