NEVANLINNA PARAMETRIZATION OF THE SOLUTIONS TO SOME RATIONAL MOMENT PROBLEMS

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Abstract. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of arbitrary (interpolation) points in $\bar{\mathbb{R}}\setminus\{0\}$, $\alpha_0 = \infty$. Consider the functions

$$b_n(z) = \frac{z^n}{\prod_{k=0}^{n}(1 - z/\alpha_k)}, \quad n = 0, 1, \ldots$$

and the moments

$$\mu_{nm} = \int b_n(t) b_m(t) d\mu(t), \quad n, m = 0, 1, \ldots.$$

We prove that, if the sequences μ_{n0}, $n = 0, 1, \ldots$, and μ_{nm}, $n, m = 0, 1, \ldots$ give rise respectively to infinitely many solutions of the associated moment problems, then these solutions may be partially described by the formula

$$\int \frac{1 + tz}{t - z} d\mu(t) = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}.$$

The four functions $A(z), B(z), C(z)$ and $D(z)$ are certain limits of quasi-orthogonal functions and φ is in the extended Nevanlinna class.

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1 Introduction

The first complete definition of the moment problem dates back to the 19th century in the work of T. Stieltjes [17]. He wrote: "We shall give the name moment problem to the following problem: It is required to find the distribution of positive mass on the interval $[0, +\infty)$, given the moments of order k $(k = 0, 1, 2, \ldots)$ of the distribution". His definition of moment of order k was the value of the integral

$$s_k = \int_0^{\infty} t^k d\sigma(t)$$

and σ was to bear the name of distribution. Stieltjes succeeded in solving this problem by introducing certain types of continued fractions. The immediate generalization to the whole real line was given by H. Hamburger in [8] where he studied the more general moments $s_k = \int_{-\infty}^{\infty} t^k d\sigma(t)$. Again, the continued fractions proved to be useful in Hamburger's proof. For a detailed exposition of different moment problems the book of N. I. Achieser [1] is an excellent reference.

Following Achieser, when posing a moment problem three fundamental questions arise:

1. Find conditions on the moment sequence s_k for the problem to have a solution.

2. Is the solution obtained unique?

3. Describe the family of solutions in case there exists more than one.

The answer to *3.* for the Hamburger moment problem is due to R. Nevanlinna, see [11]. The result may be found in Achieser's book, Th. 3.2.2 and also in [16]. Let us briefly summarize this result.

1.1 The Nevanlinna class and the parametrization of solutions

Denote by U the upper half-plane $\{\Im z > 0\}$.

DEFINITION 1. *A function φ is said to be in the Nevanlinna class $\mathcal{N}$ if it is holomorphic in U and its range of values belongs to $\bar{U}$. The extended Nevanlinna class $\mathcal{N}^*$ is the set $\mathcal{N} \cup \{\infty\}$.*

THEOREM 2 (Nevanlinna). *The formula*

$$I_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)} \tag{1}$$

establishes a one-to-one correspondence between the aggregate of solutions μ of the Hamburger moment problem and the functions φ of the extended Nevanlinna class.

The four functions $A(z), B(z), C(z)$ and $D(z)$ are certain limits of quasi-orthogonal polynomials. They are entire functions of at most minimal type of order 1. It has recently been shown by Berg and Pedersen [2] that all the four functions are of exactly the same order and type, and the order and type have been determined for various specific subclasses of moment sequences. See e.g. [9],[3],[18]. For a somewhat different approach to the Nevanlinna parametrization, see [4].

To determine $\mu(t)$ use the Stieltjes-Perron inversion formula (see [4]):

$$\begin{aligned} \mu(\{a\}) &= \lim_{s\downarrow 0} isI_\mu(a+is), \\ \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\}) + \mu(]a,b[) &= \lim_{s\downarrow 0} \int_a^b [I_\mu(t-is) - I_\mu(t+is)]\, dt. \end{aligned}$$

1.2 Statement of the problem

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of arbitrary (interpolation) points in $\bar{\mathbb{R}} \setminus \{0\}$, $\alpha_0 = \infty$ and $\pi_n(z) = \prod_{k=1}^{n} (1 - z/\alpha_k)$. Consider the functions

$$b_n(z) = \frac{z^n}{\pi_n(z)}, \quad n = 0, 1, \dots$$

and the moments

$$\mu_{nm} = \int b_n(t) b_m(t) d\mu(t), \quad n, m = 0, 1, \ldots.$$

The problems *1.* and *2.* (on the existence and uniqueness of μ) have been addressed by A. Bultheel, P. Gonzalez-Vera, E. Hendriksen and O. Njåstad [5], [6], [7]. Other references to analog problems can be found in the works of O. Njåstad [12],[13],[14], W. Jones, O. Njåstad and W.J. Thron [10], O. Njåstad and W. Thron [15]. They introduced the concept of Orthogonal Rational Function (or shortly O.R.F.) and extended many interesting well known properties from the classical theory of orthogonal polynomials.

In this paper we are interested in solving the third question. Namely, the sequences μ_{nm}, $n, m = 0, 1, \ldots$ and μ_{n0}, $n = 0, 1, \ldots$ determine two kinds of moment problems. If these problems admit infinitely many solutions, then we prove that the resulting measures are characterized by a formula like (1).

The sections 2-6 are introductory material from the theory of O.R.F. where [5] is followed very closely. The Theorem 9 is the main result of this paper.

2 Rational function spaces and ORFs

Let $\mathcal{P}_n$ denote the polynomials of degree at most n. Consider the spaces

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : \, p_n(z) \in \mathcal{P}_n \right\}, \quad n = 0, 1, \ldots$$

and put $\mathcal{L}_\infty = \cup_{n \geqslant 0} \mathcal{L}_n$, $\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_n$ and $\mathcal{R}_\infty = \mathcal{L}_\infty \cdot \mathcal{L}_\infty$.

Let M be a Hermitian, positive definite, linear functional in $\mathcal{R}_\infty$, i.e., $M\left\{\overline{f}\right\} = \overline{M\left\{f\right\}}$ for $f \in \mathcal{R}_\infty$ and $M\left\{f\overline{f}\right\} > 0$ for $f \in \mathcal{L}_\infty, f \neq 0$. This functional defines the following inner product

$$\langle f, g \rangle_M = M\left(f\bar{g}\right), \;\; f, g \in \mathcal{L}_\infty.$$

We shall additionally assume $M(1) = 1$.

The *orthogonal rational functions* $\phi_n(z)$ are by definition, the functions obtained by orthogonalizing the functions (with the Gram-Schmidt algorithm)

$$b_n(z) = \frac{z^n}{\pi_n(z)}, \quad n = 0, 1, \ldots$$

using the inner product $\langle \cdot, \cdot \rangle_M$. One has

$$\phi_n(z) = \kappa_n b_n(z) + \kappa_n' b_{n-1}(z) + \cdots + \phi_n(0). \tag{2}$$

Assume also that the *leading coefficient* κ_n is positive. Note that we recover the classical definition of orthogonal polynomials by letting all α_k be infinity.

The *Riesz-Herglotz-Nevanlinna kernel* $D(t, z)$ is given by

$$D(t, z) = \frac{1 + tz}{t - z}. \tag{3}$$

(In [5] this kernel was defined by $-i\frac{1+tz}{t-z}$, consequently some results quoted in this paper are slightly changed.)
The functions $Z_n(z)$ given by

$$Z_n(z) = \frac{z}{1 - \frac{z}{\alpha_n}}, \quad n = 0, 1, \ldots \tag{4}$$

are used in the definition of the *functions of the second kind*:

$$\psi_n(z) = M_t\left\{D(t,z)\left[\phi_n(t) - \phi_n(z)\right]\right\} - \delta_{n0} Z_0(z). \tag{5}$$

We shall say that ϕ_n, or that the index n is *regular,* if the numerator in ϕ_n does not vanish at α_{n-1}, i.e., $\phi_n(z) = \frac{p_n(z)}{\pi_n(z)}$, and $p_n(\alpha_{n-1}) \neq 0$. ($p_n(\infty) \neq 0$ means that p has degree n.) An equivalent formulation of the regularity condition is (see Lemma 11.1.5 in [5])

$$n \text{ regular} \Leftrightarrow E_n = \frac{1}{\kappa_{n-1}}\left[\kappa_n + \frac{\kappa_n'}{Z_n(\alpha_{n-1})}\right] \neq 0. \tag{6}$$

Define

$$\chi_n(z,s) = \psi_n(z) + s\phi_n(z).$$

The following theorem gives a Christoffel-Darboux type formula:

THEOREM 3. *Let* $H(z,w) = -\frac{z-w}{zw}$. *For arbitrary complex numbers* s, t *in* $\mathbb{C}$

$$\begin{aligned} & \frac{\chi_n(w,t)\chi_{n-1}(z,s)}{Z_n(w)Z_{n-1}(z)} - \frac{\chi_n(z,s)\chi_{n-1}(w,t)}{Z_n(z)Z_{n-1}(w)} \\ = \;& H(z,w)E_n\left[\sum_{k=1}^{n-1}\chi_k(z,s)\chi_k(w,t) + \left[st + 1 + D(z,w)(t-s)\right]\right]. \end{aligned} \tag{7}$$

Proof. Theorem 11.3.4, [5].

3 Rational moment problems

Let, as before, M be a Hermitian, positive definite functional in $\mathcal{R}_\infty$. Two different moment problems may be considered in the context of rational function spaces. The *moment problem in* $\mathcal{L}_\infty$ consists in finding a measure μ such that

$$\mu_{n0} = \int b_n(t)d\mu(t), \quad n = 0, 1, \ldots,$$

where μ_{n0} are certain 'moments' of the functional M, i.e.,

$$\mu_{n0} = M\left\{b_n\right\}, \quad n = 0, 1, \ldots .$$

This is the clearly equivalent to solving the representation problem

$$M\left\{f\right\} = \int f(t)d\mu(t) \tag{8}$$

for f in $\mathcal{L}_\infty$.

The *moment problem in* $\mathcal{R}_\infty$ is to find a measure μ which solves (8) in $\mathcal{R}_\infty$. That is, by knowing

$$\mu_{nm} = M\left\{b_n \bar{b}_m\right\}, \quad n, m = 0, 1, \ldots$$

one is to find μ, for which

$$\mu_{nm} = \int b_n(t)\bar{b}_m(t)d\mu(t), \quad n, m = 0, 1, \ldots .$$

We shall denote by $\mathcal{M}^{\mathcal{L}}$ ($\mathcal{M}^{\mathcal{R}}$) the set of measures solving the moment problem in $\mathcal{L}_\infty$ ($\mathcal{R}_\infty$). Evidently, $\mathcal{M}^{\mathcal{R}} \subset \mathcal{M}^{\mathcal{L}}$. If $\mathcal{R}_\infty = \mathcal{L}_\infty$ then these two sets coincide. This is the case for example if the interpolation points α_k are repeated periodically. In the classical situation they are obviously taken periodically (all are ∞).

The moment problems are classified according to how many elements there are in the sets $\mathcal{M}^{\mathcal{L}}$ ($\mathcal{M}^{\mathcal{R}}$). If $\mathcal{M}^{\mathcal{L}}$ ($\mathcal{M}^{\mathcal{R}}$) contains just one measure, the problem in $\mathcal{L}_\infty$ ($\mathcal{R}_\infty$) is said to be *determinate*; otherwise it is said to be *indeterminate*. In this paper we are concerned with indeterminate moment problems. We shall indicate a partial solution to the problem of giving a parametrization of the solutions of both moment problems.

4 Quasi-orthogonal functions

The *quasi-orthogonal functions* and the *quasi-orthogonal functions of the second kind* are defined respectively by

$$Q_n(z,\tau) = \begin{cases} \phi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z) & if\ \tau \neq \infty \\ \frac{Z_n(z)}{Z_{n-1}(z)} \phi_{n-1}(z) & if\ \tau = \infty \end{cases},$$

$$P_n(z,\tau) = \begin{cases} \psi_n(z) + \tau \frac{Z_n(z)}{Z_{n-1}(z)} \psi_{n-1}(z) & if\ \tau \neq \infty \\ \frac{Z_n(z)}{Z_{n-1}(z)} \psi_{n-1}(z) & if\ \tau = \infty \end{cases}.$$

They are natural generalizations of the quasi-orthogonal polynomials and the quasi-orthogonal polynomials of the second kind from the classical theory. They fulfill some of its well-known properties, although they do not in general satisfy orthogonality conditions. It is known, e.g., that $q_n(z,\tau)$ in

$$Q_n(z,\tau) = \frac{q_n(z,\tau)}{\pi_n(z)}$$

has all its zeros in the real line (Lemma 11.5.2, [5]).

We shall say that $\tau \in \bar{\mathbb{R}}$ is *regular* for $Q_n(z,\tau)$ when none of the points in $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a zero of $q_n(z,\tau)$. (Recall that $\alpha_0 = \infty$, $q_n(\infty,\tau) \neq 0$ means $\deg q_n(z,\tau) = n$.) $Q_n(z,\tau)$ is *regular* if both ϕ_n and τ are regular. We call τ *singular* if it is not regular.

There are actually at most $n+1$ regular values for τ (Lemma 11.5.5, [5]). We know also that (Corollary 11.5.6 in [5]), if $Q_n(z,\tau)$ is regular, then the functions $Q_n(z,\tau)$ have n simple real zeros in the complement of $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.

5 Quadrature formula

From now on, we shall assume that the functional M on $\mathcal{R}_\infty$ is given by $M\{f\} = \int f d\mu$ where μ is a probability measure. (We deal with the indeterminate moment problem.)

The quantities $\xi_i = \xi_{ni}(\tau)$, $i = 1, \ldots, n$ shall denote the n simple zeros of the regular quasi-orthogonal function $Q_n(z) = Q_n(z,\tau)$. Using the Lagrange interpolation functions

$$L_{ni}(z) = \prod_{k=1}^{n-1} \frac{\xi_i - \alpha_k}{z - \alpha_k} \prod_{k \neq i}^{n} \frac{z - \xi_k}{\xi_i - \xi_k}$$

one defines the coefficients $\lambda_{ni}(\tau) := M(L_{ni}(z))$, $i = 1, 2, \ldots, n$.

THEOREM 4. *The quadrature formula*

$$\int f d\mu = \sum_{i=1}^{n} \lambda_{ni} f(\xi_i), \tag{9}$$

with nodes at $\xi_i = \xi_{ni}(\tau)$ *and weights* $\lambda_{ni} = \lambda_{ni}(\tau)$ *, is valid in* $\mathcal{R}_{n-1}$*. The weights are positive.*

Proof. [5], Theorem 11.6.1.

This formula tells us that the atomic measure μ_n^τ defined by

$$\mu_n^\tau(x) = \sum_{i=1}^{n} \lambda_{ni} \delta_{\xi_i}(x). \tag{10}$$

where $\delta_{\xi_i}(x)$ is the Dirac delta function at $x = \xi_i$, solves the truncated moment problem in $\mathcal{R}_{n-1}$. In other words: The measures μ and μ_n^τ define the same inner product in $\mathcal{L}_{n-1}$.

6 Nested disks

Let $z \notin \mathbb{R}$ be fixed. The rational function

$$R_n(z,\tau) = -\frac{P_n(z,\tau)}{Q_n(z,\tau)}$$

as a function of τ, transforms the extended real line onto a circle $K_n(z)$ that does not degenerates into a line. Namely, for $\tau \in \bar{\mathbb{R}}$, all the zeros of $Q_n(z,\tau)$ are real. Thus for a given $z \notin \mathbb{R}$, $Q_n(z,\tau) \neq 0$ for all $\tau \in \bar{\mathbb{R}}$.

Some properties about the closed disks $\Delta_n(z)$ bounded by $K_n(z)$ are summarized in the following theorem:

THEOREM 5. *Let n be a regular index. For $z \in \mathbb{C} \setminus \mathbb{R}$:*

i) $K_n(z) = \left\{ z \in \mathbb{C} : \sum_{k=1}^{n-1} |\psi_k(z) + s\phi_k(z)|^2 + |1-s|^2 = (s+\bar{s})\frac{1+z^2}{2\Im z} \right\}$,

$\Delta_n(z)$ *is obtained by replacing* $=$ *by* $\leq$ *.*

ii) The circle $K_n(z)$ is in the upper half plane.

iii) If m and n are regular indices, $m > n$, then $\Delta_m(z) \subset \Delta_n(z)$.

iv) The circles $K_n(z)$ and $K_{n-1}(z)$ touch.

v) If n is a singular index, the whole plane is mapped to a point.

Proof. May be found in Theorem 11.7.1, [5] together with formulas for the center and the radius of $K_n(z)$.

A very important assumption is that the sequence Λ of regular indices is infinite. By *iii)* above, the discs $\Delta_n(z)$, $n \in \Lambda$ are nested and we may thus define the *limit disk* $\Delta_\infty(z) = \bigcap_{n \in \Lambda} \Delta_n(z)$. We say that we are in the *limit point case* if Δ_∞ is a point. The *limit disk case* is when Δ_∞ is a disk with positive radius.

In the following we denote by $\mathbb{C}_\alpha$ the set $\mathbb{C} \setminus \left\{ \hat{A} \cup \{-i, i\} \right\}$ where $\hat{A}$ is the closure of the interpolation points $\{\alpha_n\}_{n=1}^{\infty}$.

We have the following invariance theorem:

THEOREM 6. *Suppose that for a given* $z_0 \in \mathbb{C} \setminus \{\mathbb{R} \cup \{-i, i\}\}$, $\Delta_\infty(z_0)$ *has positive radius. Then* $\Delta_\infty(z)$ *has positive radius for every* $z \in \mathbb{C} \setminus \{\mathbb{R} \cup \{-i, i\}\}$. *Furthermore the series*

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 \quad \text{and} \quad \sum_{k=0}^{\infty} |\psi_k(z)|^2 \tag{11}$$

converge locally uniformly in $\mathbb{C}_\alpha$.

Proof. [5], Lemma 11.7.3 and Theorem 11.7.5. The convergence of the series (11) is formulated for $z \in \mathbb{C} \setminus \{\mathbb{R} \cup \{-i, i\}\}$, but the argument implies convergence for $z \in \mathbb{C}_\alpha$.

7 Riesz-Herglotz-Nevanlinna transform and limit circle

It was proved in [5], Theorem 11.8.1 that $\mathcal{M}^{\mathcal{L}}$ (the set of solutions of the moment problem in $\mathcal{L}_\infty$) is not empty provided that there are infinitely many regular indices associated to the sequence $\{\phi_n\}$. In this case, we have the following description of the value sets of the *Riesz-Herglotz-Nevanlinna transform*

$$\Omega_\mu(z) = \int \frac{1 + tz}{t - z} d\mu$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, in terms of the limit circle:

THEOREM 7. $\left\{\Omega_\mu(z) : \mu \in \mathcal{M}^{\mathcal{R}}\right\} \subset \Delta_\infty(z) \subset \left\{\Omega_\mu(z) : \mu \in \mathcal{M}^{\mathcal{L}}\right\}$.

Proof. [5], Theorem 11.8.2.

Note that if $\mathcal{R}_\infty = \mathcal{L}_\infty$, the inclusions may be replaced by equalities. This is well known from the classical situation, see [1], Theorem 2.2.4.

We have the following consequence of the first inclusion:

COROLLARY 8. *If* $\mathcal{M}^{\mathcal{R}} \neq \emptyset$ *and at least one of the series in (11) diverges, then the moment problem in* $\mathcal{R}_\infty$ *is determinate.*

8 The functions A_n, B_n, C_n, D_n

Let $x_0 \notin \{0, \alpha_1, \ldots, \alpha_n, \ldots\}$ be a fixed real number. Define the following rational functions

$$A_n(z, x_0) = E_n H(z, x_0) \left[\sum_{k=1}^{n-1} \psi_k(z)\psi_k(x_0) + 1\right], \tag{12}$$

$$B_n(z, x_0) = -E_n H(z, x_0) \left[\sum_{k=1}^{n-1} \phi_k(z)\psi_k(x_0) - D(z, x_0)\right], \tag{13}$$

$$C_n(z, x_0) = E_n H(z, x_0) \left[\sum_{k=1}^{n-1} \psi_k(z)\phi_k(x_0) + D(z, x_0)\right], \tag{14}$$

$$D_n(z, x_0) = E_n H(z, x_0) \left[\sum_{k=1}^{n-1} \phi_k(z)\phi_k(x_0) + 1\right]. \tag{15}$$

Then by the Christoffel-Darboux relations for $(s,t) \in \{(0,0),(0,\infty),(\infty,0),(\infty,\infty)\}$ (by, e.g., $(s,t)=(\infty,\infty)$ we mean to divide (7) by st and let s,t tend to infinity)

$$A_n(z,x_0) = \left[\hat{\psi}_n(x_0)\hat{\psi}_{n-1}(z) - \hat{\psi}_n(z)\hat{\psi}_{n-1}(x_0)\right], \tag{16}$$

$$B_n(z,x_0) = \left[\hat{\psi}_n(x_0)\hat{\phi}_{n-1}(z) - \hat{\phi}_n(z)\hat{\psi}_{n-1}(x_0)\right], \tag{17}$$

$$C_n(z,x_0) = \left[\hat{\phi}_n(x_0)\hat{\psi}_{n-1}(z) - \hat{\psi}_n(z)\hat{\phi}_{n-1}(x_0)\right], \tag{18}$$

$$D_n(z,x_0) = \left[\hat{\phi}_n(x_0)\hat{\phi}_{n-1}(z) - \hat{\phi}_n(z)\hat{\phi}_{n-1}(x_0)\right], \tag{19}$$

where we have written $\hat{\phi}_n(\cdot), \hat{\psi}_n(\cdot)$ instead of $\frac{\phi_n(\cdot)}{Z_n(\cdot)}$ and $\frac{\psi_n(\cdot)}{Z_n(\cdot)}$.

The functions A_n, B_n,C_n and D_n will play a fundamental role throughout this paper. In Section 10 we show that these functions define, by letting n tend to infinity, four functions A, B,C and D, analytic in $\mathbb{C}_\alpha$.

9 The Nevanlinna Parametrization

We may now state the main theorem of this paper.

THEOREM 9. *Assume that the moment problem in $\mathcal{R}_\infty$ is indeterminate and that the sequence of regular indices contains infinitely many elements. Then there exist functions A,B,C,D, analytic in $\mathbb{C}_\alpha$, such that*

i) For each function φ in the extended Nevanlinna class, there exists a solution μ of the moment problem in $\mathcal{L}_\infty$ such that

$$\int \frac{1+tz}{t-z} d\mu(t) = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \tag{20}$$

ii) For each solution μ of the moment problem in $\mathcal{R}_\infty$, there is a function φ in the extended Nevanlinna class such that (20) holds.

REMARK 10. *ii) is still valid in the more general situation where μ is any solution of the moment problem in $\mathcal{L}_\infty$ satisfying $\int \frac{1+tz}{t-z} d\mu(t) \in \Delta_\infty(z)$ for all z.*

REMARK 11. *In case we have $\mathcal{L}_\infty = \mathcal{R}_\infty$, the theorem implies the existence of a one-to-one correspondence between Nevanlinna functions and solutions of the moment problem, like in the classical situation.*

10 Auxiliary results

In the following K stands for a compact set in $\mathbb{C}_\alpha = \mathbb{C} \setminus \left\{\hat{A} \cup \{-i,i\}\right\}$ where $\hat{A}$ is the closure of $\{\alpha_n\}_{n=1}^\infty$.

PROPOSITION 12. *Let $z \in \mathbb{C}_\alpha$ and let n be a regular index. The following assertions hold for $R_n(z,\tau) = -\frac{P_n(z,\tau)}{Q_n(z,\tau)}$ and A_n, B_n, C_n, D_n defined as in (12)-(15):*

a) $(A_nD_n - B_nC_n)(z,x_0) = E_n^2 \left(\frac{1+z^2}{z^2}\right)\left(\frac{1+x_0^2}{x_0^2}\right)$ for $z \in \mathbb{C}_\alpha \setminus \{0\}$.

b) For

$$t = t_n(x_0,\tau) = \begin{cases} \frac{\hat{\phi}_n(x_0)+\tau\hat{\phi}_{n-1}(x_0)}{\hat{\psi}_n(x_0)+\tau\hat{\psi}_{n-1}(x_0)} & if\ \tau \neq \infty \\ \frac{\hat{\phi}_{n-1}(x_0)}{\hat{\psi}_{n-1}(x_0)} & if\ \tau = \infty \end{cases} \tag{21}$$

one has

$$R_n(z,\tau) = \begin{cases} -\frac{A_n(z,x_0)t-C_n(z,x_0)}{B_n(z,x_0)t-D_n(z,x_0)} & if\ t \neq \infty \\ -\frac{A_n(z,x_0)}{B_n(z,x_0)} & if\ t = \infty \end{cases}. \tag{22}$$

c) If the limit disk case occurs and the sequence Λ of regular indices is infinite, then there exist functions $A(z,x_0)$, $B(z,x_0)$, $C(z,x_0)$ and $D(z,x_0)$, analytic in $\mathbb{C}_\alpha$, such that the following limiting relationships hold uniformly in K for fixed t:

$$\frac{1}{E_n}A_n(z,x_0) \stackrel{n\in\Lambda}{\longrightarrow} A(z,x_0),$$

$$\frac{1}{E_n}B_n(z,x_0) \stackrel{n\in\Lambda}{\longrightarrow} B(z,x_0),$$

$$\frac{1}{E_n}C_n(z,x_0) \stackrel{n\in\Lambda}{\longrightarrow} C(z,x_0),$$

$$\frac{1}{E_n}D_n(z,x_0) \stackrel{n\in\Lambda}{\longrightarrow} D(z,x_0).$$

Furthermore,

$$R_n(z,\tau) \stackrel{n\in\Lambda}{\longrightarrow} \begin{cases} -\frac{A(z,x_0)t-C(z,x_0)}{B(z,x_0)t-D(z,x_0)} & if\ t \neq \infty \\ -\frac{A(z,x_0)}{B(z,x_0)} & if\ t = \infty \end{cases}$$

uniformly in K for t fixed, provided $K \subset \mathbb{C} \setminus \{\mathbb{R} \cup \{-i,i\}\}$.

Proof.

$$\begin{aligned} & a)\ (A_nD_n - B_nC_n)(z,x_0) \\ = & \left[\hat{\psi}_n(x_0)\hat{\psi}_{n-1}(z) - \hat{\psi}_n(z)\hat{\psi}_{n-1}(x_0)\right]\left[\hat{\phi}_n(x_0)\hat{\phi}_{n-1}(z) - \hat{\phi}_n(z)\hat{\phi}_{n-1}(x_0)\right] \\ & - \left[\hat{\psi}_n(x_0)\hat{\phi}_{n-1}(z) - \hat{\phi}_n(z)\hat{\psi}_{n-1}(x_0)\right]\left[\hat{\phi}_n(x_0)\hat{\psi}_{n-1}(z) - \hat{\psi}_n(z)\hat{\phi}_{n-1}(x_0)\right] \\ = & -\hat{\psi}_n(x_0)\hat{\psi}_{n-1}(z)\hat{\phi}_n(z)\hat{\phi}_{n-1}(x_0) - \hat{\psi}_n(z)\hat{\psi}_{n-1}(x_0)\hat{\phi}_n(x_0)\hat{\phi}_{n-1}(z) \\ & +\hat{\psi}_n(x_0)\hat{\phi}_{n-1}(z)\hat{\psi}_n(z)\hat{\phi}_{n-1}(x_0) + \hat{\phi}_n(z)\hat{\psi}_{n-1}(x_0)\hat{\phi}_n(x_0)\hat{\psi}_{n-1}(z), \end{aligned}$$

which factors as $F_n(z,z)F_n(x_0,x_0)$ with

$$F_n(z,z) = \hat{\psi}_n(z)\hat{\phi}_{n-1}(z) - \hat{\phi}_n(z)\hat{\psi}_{n-1}(z).$$

The conclusion follows by the determinant formula (see [5, Th.11.2.3])

$$F_n(z,z) = \frac{1+z^2}{z^2}E_n, \tag{23}$$

and the fact that $E_n \neq 0$ iff n is a regular index.

b) Assume first $\tau \neq \infty$ and $t \neq \infty$. By the definition of t, and B_n and D_n given by (17) and (19) respectively,

$$\begin{aligned} & B_n(z,x_0)t - D_n(z,x_0) \\ = \; & \frac{1}{\hat\psi_n(x_0) + \tau\,\hat\psi_{n-1}(x_0)} \Big[\Big(\hat\psi_n(x_0)\hat\phi_{n-1}(z) - \hat\phi_n(z)\hat\psi_{n-1}(x_0) \Big) \Big(\hat\phi_n(x_0) + \tau\hat\phi_{n-1}(x_0) \Big) \\ & - \Big(\hat\phi_n(x_0)\hat\phi_{n-1}(z) - \hat\phi_n(z)\hat\phi_{n-1}(x_0) \Big) \Big(\hat\psi_n(x_0) + \tau\hat\psi_{n-1}(x_0) \Big) \Big], \end{aligned}$$

which reduces to

$$\begin{aligned} & \frac{1}{\hat\psi_n(x_0) + \tau\,\hat\psi_{n-1}(x_0)} \Big[\hat\phi_n(z) \Big(\hat\psi_n(x_0)\hat\phi_{n-1}(x_0) - \hat\phi_n(x_0)\hat\psi_{n-1}(x_0) \Big) \\ & + \tau\hat\phi_{n-1}(z) \Big(\hat\psi_n(x_0)\hat\phi_{n-1}(x_0) - \hat\phi_n(x_0)\hat\psi_{n-1}(x_0) \Big) \Big] \\ = \; & \frac{1}{\hat\psi_n(x_0) + \tau\,\hat\psi_{n-1}(x_0)} \Big(\hat\psi_n(x_0)\hat\phi_{n-1}(x_0) - \hat\phi_n(x_0)\hat\psi_{n-1}(x_0) \Big) \Big(\hat\phi_n(z) + \tau\hat\phi_{n-1}(z) \Big). \end{aligned}$$

By the determinant formula ([5, Th.11.2.3]) and the definition of Q_n

$$B_n(z,x_0)t - D_n(z,x_0) = \frac{1}{\hat\psi_n(x_0) + \tau\,\hat\psi_{n-1}(x_0)} E_n \left(\frac{1+x_0^2}{x_0^2} \right) \frac{Q_n(z,\tau)}{Z_n(z)}. \tag{24}$$

Analogous calculations show that

$$A_n(z,x_0)t - C_n(z,x_0) = \frac{1}{\hat\psi_n(x_0) + \tau\,\hat\psi_{n-1}(x_0)} E_n \left(\frac{1+x_0^2}{x_0^2} \right) \frac{P_n(z,\tau)}{Z_n(z)}. \tag{25}$$

Dividing (25) by (24) we get (22) provided that E_n does not vanish, which in turn follows from the assumption that n is a regular index (see (6)).

The case $\tau = \infty$, $t \neq \infty$ is simpler:

$$\begin{aligned} & B_n(z,x_0)t - D_n(z,x_0) \\ = \; & \frac{1}{\hat\psi_{n-1}(x_0)} \Big[\Big(\hat\psi_n(x_0)\hat\phi_{n-1}(z) - \hat\phi_n(z)\hat\psi_{n-1}(x_0) \Big) \hat\phi_{n-1}(x_0) \\ & - \Big(\hat\phi_n(x_0)\hat\phi_{n-1}(z) - \hat\phi_n(z)\hat\phi_{n-1}(x_0) \Big) \hat\psi_{n-1}(x_0) \Big] \\ = \; & \frac{1}{\hat\psi_{n-1}(x_0)} \Big[\hat\phi_{n-1}(z) \Big(\hat\psi_n(x_0)\hat\phi_{n-1}(x_0) - \hat\phi_n(x_0)\hat\psi_{n-1}(x_0) \Big) \Big] \\ = \; & \frac{1}{\hat\psi_{n-1}(x_0)} E_n \left(\frac{1+x_0^2}{x_0^2} \right) \frac{Q_n(z,\tau)}{Z_n(z)}. \end{aligned}$$

Analogously

$$A_n(z,x_0)t - C_n(z,x_0) = \frac{1}{\hat\psi_{n-1}(x_0)} E_n \left(\frac{1+x_0^2}{x_0^2} \right) \frac{P_n(z,\tau)}{Z_n(z)},$$

and again (22) follows.

It remains to see what happens if $t = \infty$. Two possibilities arise: if $\tau = \infty$, then $\hat\psi_{n-1}(x_0) = 0$ ($\hat\phi_{n-1}(x_0) \neq 0$ and $\hat\psi_n(x_0) \neq 0$ by the determinant formula), $A_n(z,x_0) = \hat\psi_n(x_0)\hat\psi_{n-1}(z)$, $B_n(z,x_0) = \hat\psi_n(x_0)\hat\phi_{n-1}(z)$ and

$$R_n(z,\infty) = -\frac{\hat\psi_{n-1}(z)}{\hat\phi_{n-1}(z)} = -\frac{A_n(z,x_0)}{B_n(z,x_0)}. \tag{26}$$

If $\tau \neq \infty$ then $\hat{\psi}_{n-1}(x_0) \neq 0$ and so $\tau = -\frac{\hat{\psi}_n(x_0)}{\hat{\psi}_{n-1}(x_0)}$, thus

$$R_n(z,\tau) = -\frac{\hat{\psi}_n(z) + \tau\hat{\psi}_{n-1}(z)}{\hat{\phi}_n(z) + \tau\hat{\phi}_{n-1}(z)} = -\frac{\hat{\psi}_n(z)\hat{\psi}_{n-1}(x_0) - \hat{\psi}_n(x_0)\hat{\psi}_{n-1}(z)}{\hat{\phi}_n(z)\hat{\psi}_{n-1}(x_0) - \hat{\psi}_n(x_0)\hat{\phi}_{n-1}(z)}$$

$$(27) \qquad = -\frac{A_n(z,x_0)}{B_n(z,x_0)}.$$

c) As stated in Theorem 6, if $\Delta_\infty(z)$ has positive radius, then

$$(28) \qquad \sum_{k=0}^{\infty} \mid \phi_k(z) \mid^2 < \infty,$$

and

$$(29) \qquad \sum_{k=0}^{\infty} \mid \psi_k(z) \mid^2 < \infty,$$

hold uniformly in K.

By definitions (12)-(15), the Cauchy-Schwartz inequality and (28)-(29), we have e.g., that the sequence $\frac{B_n}{E_n}$ is a Cauchy sequence. Namely, for $n > m > 0$,

$$\left| \frac{B_n}{E_n} - \frac{B_m}{E_n} \right| = \left| H(z,x_0) \left[\sum_{k=m}^{n-1} \phi_k(z)\psi_k(x_0) \right] \right|$$

$$\leq |H(z,x_0)| \sum_{k=m}^{n-1} |\phi_k(z)\psi_k(x_0)| \leq |H(z,x_0)| \left(\sum_{k=m}^{n-1} |\phi_k(z)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=m}^{n-1} |\psi_k(x_0)|^2 \right)^{\frac{1}{2}} \overset{n,m}{\longrightarrow} 0,$$

uniformly in K. This proves uniform convergence on compact subsets of the sequence $\frac{1}{E_n}B_n$ to a holomorphic function in $\mathbb{C}_\alpha$, giving the first four asymptotics.

If we write (22) as

$$R_n(z,\tau) = \begin{cases} -\dfrac{\frac{A_n(z,x_0)}{E_n}t - \frac{C_n(z,x_0)}{E_n}}{\frac{B_n(z,x_0)}{E_n}t - \frac{D_n(z,x_0)}{E_n}} & if\ t \neq \infty \\ -\dfrac{\frac{A_n(z,x_0)}{E_n}}{\frac{B_n(z,x_0)}{E_n}} & if\ t = \infty \end{cases}$$

and we let t be fixed and n tend to infinity along the sequence of regular indices Λ,

$$R_n(z,\tau) \overset{n\in\Lambda}{\longrightarrow} \begin{cases} -\frac{A(z,x_0)t - C(z,x_0)}{B(z,x_0)t - D(z,x_0)} & if\ t \neq \infty \\ -\frac{A(z,x_0)}{B(z,x_0)} & if\ t = \infty \end{cases}.$$

Note e.g., that the limit functions $A(z,x_0)t - C(z,x_0)$ and $B(z,x_0)t - D(z,x_0)$ cannot vanish simultaneously, otherwise

$$A(z,x_0)D(z,x_0) - B(z,x_0)C(z,x_0) = 0,$$

contradicting $a)$ for $n \to \infty$. To prove uniform convergence, we must show that for $z \in K$, the expressions $\frac{B_n(z,x_0)}{E_n}t - \frac{D_n(z,x_0)}{E_n}$, $B(z,x_0)t - D(z,x_0)$ for $t \neq \infty$ and $\frac{B_n(z,x_0)}{E_n}$, $B(z,x_0)$ for $t = \infty$ are uniformly bounded away from zero. Take for example the case $t \neq \infty$. The denominator in $R_n(z,\tau)$ does not have any zero outside $\mathbb{R}$. Hence, by Hurwitz theorem, the limit function $B(z,x_0)t - D(z,x_0)$ cannot have any zero outside the real line either. It follows by continuity of $\Im(\cdot)$ that there is an $\epsilon > 0$ such that

$$|\Im(B(z,x_0)t - D(z,x_0))| > \epsilon,$$

uniformly in K. Finally, by the uniform convergence of the denominators of $R_n(z,\tau)$ in K, there is a positive η such that $\left|\Im\left(\frac{B_n(z,x_0)}{E_n}t - \frac{D_n(z,x_0)}{E_n}\right)\right| > \eta$ for all $z \in K$ and n large enough.

LEMMA 13. *Let n be a regular index. Define the rational functions*

$$w_n(t) = -\frac{A_n(z,x_0)t - C_n(z,x_0)}{B_n(z,x_0)t - D_n(z,x_0)},$$

$$w(t) = -\frac{A(z,x_0)t - C(z,x_0)}{B(z,x_0)t - D(z,x_0)}.$$

For fixed $z \in U$, w_n and w transform $\bar{U}$ into $\Delta_n(z)$, and $\bar{U}$ into $\Delta_\infty(z)$ respectively. These correspondences are bijective.

Proof. It follows by the formula in Proposition 12 $a)$ that

$$\frac{dw_n(t)}{dt} = \frac{E_n^2\left(\frac{1+z^2}{z^2}\right)\left(\frac{1+x_0^2}{x_0^2}\right)}{(B_n(z,x_0)t - D_n(z,x_0))^2},$$

which is positive if $z \in \mathbb{R} \setminus \left\{\hat{A} \cup \{0\}\right\}$. In this situation $w_n(t)$ increases along $\mathbb{R}$, as t increases along $\mathbb{R}$, and so w_n is an automorphism of the upper half-plane. Take z close to the real axis. It is known from Proposition 12 and Theorem 5 that w_n transforms the real line into a circle contained in U. By continuity, this circle must be oriented in the counter clock-wise sense. Hence, w_n maps $\bar{U}$ into the disc $\Delta_n(z)$. This may be used to prove that w cannot transform U into the exterior of $\Delta_\infty(z)$. Indeed, let $t \in U$. Since $w(t)$ is the point-wise limit of the sequence $w_n(t) \in \Delta_n(z)$, and these discs form a nested sequence associated to consecutive regular indices (see Theorem 5), then $w(t) \in \bigcap_{n\in\Lambda} \Delta_n(z) = \Delta_\infty(z)$, where Λ is the sequence of regular indices.

The last result of this section deals with the question of how x_0 must be chosen to achieve the regularity of $B_n(z,x_0)$ which we make use of later.

We have

$$B_n(z,x_0) = \begin{cases} -\frac{\hat{\psi}_{n-1}(x_0)}{Z_n(z)}Q_n(z,\tau_n) & if\ \hat{\psi}_{n-1}(x_0) \neq 0 \\ -\frac{\hat{\psi}_n(x_0)}{Z_n(z)}Q_n(z,\tau_n) & if\ \hat{\psi}_{n-1}(x_0) = 0 \end{cases}$$

for

$$\tau_n = \begin{cases} -\frac{\hat{\psi}_n(x_0)}{\hat{\psi}_{n-1}(x_0)} & if\ \hat{\psi}_{n-1}(x_0) \neq 0 \\ \infty & if\ \hat{\psi}_{n-1}(x_0) = 0 \end{cases}.$$

Let $q_n(z,\tau_n)$ be the numerator of $Q_n(z,\tau_n)$. Write $q_n(z,x_0) = q_n(z,\tau_n,x_0)$ to emphasize its dependency on x_0 and recall that $q_n(\infty,x_0) \neq 0$ is interpreted by saying that $\deg q_n(z,\tau_n) = n$.

LEMMA 14. *We may choose* x_0 *such that*

i) τ_n *is regular for all* n, *i.e.,* $q_n(\alpha_k, x_0) \neq 0$ *for* $k = 1, 2, \ldots, n$ *for all* n,

ii) $\hat{\psi}_n(x_0) \neq 0$ *for all* n.

Proof. Let $\phi_n(z) = \frac{p_n(z)}{\pi_n(z)}$ and $\psi_n(z) = \frac{q_n(z)}{\pi_n(z)}$ be the n-th orthogonal rational function and the function of second kind respectively. We first choose x_0 outside the countable set of all the zeros of all the functions $\hat{\psi}_n(z)$, $n = 1, 2, \ldots$. Two possibilities must be considered:

Case 1: $\alpha_k \neq \infty$ for $k = 1, 2, \ldots, n$.

We may write

$$Q_n(z, \tau_n) = \phi_n(z) + \tau_n \frac{(1 - \frac{z}{\alpha_{n-1}})}{(1 - \frac{z}{\alpha_n})} \phi_{n-1}(z). \tag{30}$$

Hence

$$q_n(z, \tau_n) \ = p_n(z) + \tau_n (1 - \frac{z}{\alpha_{n-1}}) p_{n-1}(z).$$

Clearly from the definition, τ_n satisfies the equation

$$q_n(x_0) + \tau_n (1 - \frac{x_0}{\alpha_{n-1}}) q_{n-1}(x_0) = 0.$$

Now fix n. There are at most n values $\tau^{(1)}, \ldots, \tau^{(n)}$ of τ which are singular values, i.e., for which

$$p_n(\alpha_k) + \tau (1 - \frac{\alpha_k}{\alpha_{n-1}}) p_{n-1}(\alpha_k) = 0 \tag{31}$$

for $k = 1, \ldots, n$. For each such singular value $\tau^{(m)}$ there are (at most) n values $\eta_1^{(m)}, \ldots, \eta_n^{(m)}$ of x_0 such that

$$q_n(x_0) + \tau^{(m)} (1 - \frac{x_0}{\alpha_{n-1}}) q_{n-1}(x_0) = 0.$$

Let $x_0 \notin \left\{ \eta_1^{(1)}, \ldots, \eta_n^{(1)}, \ldots, \eta_1^{(n)}, \ldots, \eta_n^{(n)} \right\}$. Then the value τ_n determined by

$$q_n(x_0) + \tau_n (1 - \frac{x_0}{\alpha_{n-1}}) q_{n-1}(x_0) = 0 \tag{32}$$

is not a singular value, i.e.,

$$p_n(\alpha_k) + \tau_n (1 - \frac{\alpha_k}{\alpha_{n-1}}) p_{n-1}(\alpha_k) \neq 0$$

for $k = 1, 2, \ldots, n$.

In other words

$$q_n(\alpha_k, x_0) \neq 0 \tag{33}$$

for $k = 1, 2, \ldots, n$.

Case 2: $\alpha_k = \infty$ for some $k \in \{1, 2, \ldots, n\}$.

If $\alpha_{n-1} = \infty$ then $\deg q_n(z, \tau_n) \ = \deg p_n(z) = n$ independent of τ. If $\alpha_{n-1} \neq \infty$, there is at most one value $\tau^{(0)}$ of τ such that

$$\deg q_n(z, \tau_n) \ = \deg(p_n(z) + \tau (1 - \frac{z}{\alpha_{n-1}}) p_{n-1}(z)) < n.$$

On the other hand, there is at most one value $\eta^{(0)}$ of x_0 for which

$$q_n(x_0)+\tau^{(0)}(1-\frac{x_0}{\alpha_{n-1}})q_{n-1}(x_0)=0.$$

Let $x_0 \neq \eta^{(0)}$. Then the value τ_n determined by (32) is such that $\deg q_n(z,\tau_n)=n$.

Now let $x_0 \notin \bigcup_n \left\{\eta^{(0)},\eta_1^{(1)},\dots,\eta_n^{(1)},\dots,\eta_1^{(n)},\dots,\eta_n^{(n)}\right\} \cup \bigcup_n \left\{zeros\ of\ \hat{\psi}_n\right\}$. Then (33) and $\deg q_n(z,\tau_n)=n$ clearly hold for all n.

11 Proof of the Main Theorem

Let us first prove *ii*).

Define φ by the formula

$$\varphi(z)=\frac{\Omega_\mu(z)D(z)+C(z)}{\Omega_\mu(z)B(z)+A(z)}. \tag{34}$$

where $\Omega_\mu(z)=\int\frac{1+tz}{t-z}d\mu(t)$. If $\Omega_\mu(z)\equiv-\frac{A(z)}{B(z)}$, then $\varphi(z)\equiv\infty$ and (20) is correct. Next assume $\Omega_\mu(z)\not\equiv\frac{A(z)}{B(z)}$. It follows that φ defined by (34) is meromorphic in U. To prove that φ is in the Nevanlinna class we proceed as follows: By Theorem 7 we know that $\Omega_\mu(z)\in\Delta_\infty(z)$ (the limit disk) provided that μ is a solution of the moment problem in $\mathcal{R}_\infty$. In addition, by Lemma 13, $t\in\bar{U}$ if and only if $w(t)\in\Delta_\infty(z)$. For the value $t=\varphi(z)$ we have $w(t)=\Omega_\mu(z)\in\Delta_\infty(z)$, hence $t=\varphi(z)\in\bar{U}$ for a given $z\in U$. This property implies that φ is holomorphic and maps U into $\bar{U}$.

i) Let us now prove that for φ in the extended Nevanlinna class there exists a measure μ in $\mathcal{M}^{\mathcal{L}}$ for which (20) holds.

Consider the function

$$\zeta_n(z)=-\frac{A_n(z)\varphi(z)-C_n(z)}{B_n(z)\varphi(z)-D_n(z)}. \tag{35}$$

Let $\varphi\in\mathcal{N}^*$. Then either $\varphi(z)=t$ for some $t\in\bar{\mathbb{R}}$ or φ maps the open upper half-plane U into U. In the first case $w_n(\varphi(z))=w_n(t)\in K_\infty(z)$ for all $z\in U$. In the second case $w_n(\varphi(z))$ is in the interior of $\Delta_\infty(z)$ for all $z\in U$. Hence $\zeta_n(z)$ is actually in $\mathcal{N}$.

According to Nevanlinna's formula (see [1, §3, eq. (3.3)], p.92), there is a measure μ_n for which

$$\zeta_n(z)=az+b+\int\frac{1+tz}{t-z}d\mu_n(t)$$

holds with $a\geq 0$ and $b\in\mathbb{R}$.

By Corollary 11.7.2 in [5], for $z=i,-i$, $w_n(t)$ reduces to the points $i,-i$ respectively. It follows by letting $z=i$ in the above equation that

$$\zeta_n(z)=az+\int\frac{1+tz}{t-z}d\mu_n(t),$$

where $a=1-\int d\mu_n(t)=1-|\mu_n|$.

We shall show that $a=0$. Note that for $z\in U$

$$\frac{\zeta_n(z)}{z}=a+\int\frac{1+tz}{z(t-z)}d\mu_n(t)=a+|\mu_n|+\frac{1+z^2}{z^2}\int\frac{z}{t-z}d\mu_n(t)$$

and

$$\int \left| \frac{z}{t-z} \right| d\mu_n(t) \le \frac{|z|}{|\Im z|} |\mu_n| .$$

The above inequality shows that e.g., when $z = iy$, $y > 0$ ($\frac{|z|}{|\Im z|}$ is bounded in any sector $\delta < \varphi < \pi - \delta$, $\delta \in (0, \pi/2)$), the functions $\frac{1+tz}{z(t-z)} \in L^1(\mu_n)$ (μ_n is a finite measure as follows from the positivity of a) and that they are dominated by an integrable function. By Lebesgue's theorem $\frac{\zeta_n(z)}{z} \overset{y\to\infty}{\longrightarrow} a$. The boundedness of $\zeta_n(z)$ in the upper half-plane (see Lemma 13) implies $a = 0$, so that

$$\zeta_n(z) = \int \frac{1+tz}{t-z} d\mu_n(t). \tag{36}$$

Let us show that μ_n is a solution of a certain truncated moment problem. Using the quadrature formula in Theorem 4, it was proved in Lemma 11.10.6, [5] that for τ regular, the following expression for $R_n(z,\tau)$ holds:

$$R_n(z,\tau) = \sum_{k=1}^{n} \lambda_{nk}(\tau) \frac{1+\xi_{nk}(\tau) z}{\xi_{nk}(\tau) - z}. \tag{37}$$

Here $\xi_{nk}(\tau)$ are the n simple zeros of $Q_n(z,\tau)$.

Let τ be such that $R_n(z,\tau) = -\frac{A_n(z,x_0)}{B_n(z,x_0)}$ (i.e., $\tau = \tau_n = -\frac{\hat{\psi}_n(x_0)}{\hat{\psi}_{n-1}(x_0)}$ by (27)), take x_0 as in Lemma 14 (to ensure that $\tau = \tau_n$ is regular for $Q_n(z,\tau)$) and μ_n^τ as in (10). From (37) follows that

$$\int \frac{1+tz}{t-z} d\mu_n^\tau(t) = -\frac{A_n(z,x_0)}{B_n(z,x_0)}. \tag{38}$$

We shall now prove that μ_n and μ_n^τ define the same inner product in $\mathcal{L}_{n-1}$.

Recall that $M_\mu(f) = \int f d\mu$ gives rise to an inner product which will be denoted by $\langle \cdot,\cdot \rangle_\mu$. Let μ and ν be two positive measures on $\mathbb{R}$. The inner products $\langle \cdot,\cdot \rangle_\mu$ and $\langle \cdot,\cdot \rangle_\nu$ coincide in $\mathcal{L}_{n-1}$ iff

$$\lim_{z\to\alpha} \left[(\Omega_\mu(z) - \Omega_\nu(z))^{(k)} \right] = 0, \quad k = 0, 1, \ldots \alpha^\# - 1, \tag{39}$$

where $\alpha^\#$ is the multiplicity of α in the set $\tilde{A}_{n-1} = \{i, -i, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots \alpha_{n-1}, \alpha_{n-1}\}$.

This assertion and its proof are given in [5], Corollary 11.10.4. The limit above is in the sense that z approaches α throughout a vertical line. If α is infinity, then let $z = iy$, $y \to \infty$. We take $\lim\limits_{z\to\infty} f(z) = \lim\limits_{z\to 0} f(\frac{1}{z})$.

From (36) and (38) follows that

$$\begin{aligned} & \Omega_{\mu_n}(z) - \Omega_{\mu_n^\tau}(z) \\ = \; & -\frac{A_n(z,x_0)\varphi(z) - C_n(z,x_0)}{B_n(z,x_0)\varphi(z) - D_n(z,x_0)} + \frac{A_n(z,x_0)}{B_n(z,x_0)}. \end{aligned} \tag{40}$$

By Lemma 14, x_0 may be chosen such that $\hat{\psi}_{n-1}(x_0) \neq 0$. Thus, by Proposition 12 *a)*, (40) equals

$$\frac{-E_n^2 \left(\frac{1+z^2}{z^2}\right)\left(\frac{1+x_0^2}{x_0^2}\right)}{B_n(z,x_0)\left(B_n(z,x_0)\varphi(z) - D_n(z,x_0)\right)}. \tag{41}$$

We have

$$B_n(z,x_0) = -\frac{\hat{\psi}_{n-1}(x_0)}{Z_n(z)}Q_n(z,\tau_n)$$
$$= -\frac{\hat{\psi}_{n-1}(x_0)q_n(z,\tau_n)}{z\pi_{n-1}(z)} \tag{42}$$

where $q_n(z,\tau_n)$ is the numerator of $Q_n(z,\tau)$ and $\tau_n = -\frac{\hat{\psi}_n(x_0)}{\hat{\psi}_{n-1}(x_0)}$.

For $D_n(z,x_0)$ the situation is similar:

$$D_n(z,x_0) = \begin{cases} -\frac{\hat{\phi}_{n-1}(x_0)q_n(z,\tau_n^*)}{z\pi_{n-1}(z)} & if\ \hat{\phi}_{n-1}(x_0) \neq 0 \\ \hat{\phi}_n(x_0)\frac{p_{n-1}(z)}{z\pi_{n-2}(z)} & if\ \hat{\phi}_{n-1}(x_0) = 0 \end{cases} \tag{43}$$

where $p_{n-1}(z)$ is the numerator of $\phi_{n-1}(z)$ and $\tau_n^* = -\frac{\hat{\phi}_n(x_0)}{\hat{\phi}_{n-1}(x_0)}$.

It is not difficult to check that for $\hat{\phi}_{n-1}(x_0) \neq 0$, (41) is equal to

$$= -\frac{\pi_{n-1}^2(z)(1+z^2)E_n^2\left(\frac{1+x_0^2}{x_0^2}\right)}{\hat{\psi}_{n-1}(x_0)q_n(z,\tau_n)\left[\hat{\psi}_{n-1}(x_0)q_n(z,\tau_n)\,\varphi(z) - \hat{\phi}_{n-1}(x_0)q_n(z,\tau_n^*)\right]} \tag{44}$$

and for $\hat{\phi}_{n-1}(x_0) = 0$, (41) is equal to

$$= -\frac{\pi_{n-1}^2(z)(1+z^2)E_n^2\left(\frac{1+x_0^2}{x_0^2}\right)}{\hat{\psi}_{n-1}(x_0)q_n(z,\tau_n)\left[\hat{\psi}_{n-1}(x_0)q_n(z,\tau_n)\,\varphi(z) + \hat{\phi}_n(x_0)(1-\frac{z}{\alpha_{n-1}})p_{n-1}(z)\right]}. \tag{45}$$

We introduce the following subclasses of Nevanlinna functions:

$$\mathcal{N}_n = \left\{\varphi \in \mathcal{N} : \begin{cases} \lim_{z\to\alpha_k}(z-\alpha_k)\varphi(z) \neq 0 & if\ \alpha_k \neq \infty \\ \lim_{z\to\alpha_k} z^{-1}\varphi(z) \neq 0 & if\ \alpha_k = \infty \end{cases},\ k = 1,2,\ldots,n-1\right\}.$$

<u>**LEMMA**</u> **15.** *For $n \geq 2$, $\mathcal{N}_n$ is dense in $\mathcal{N}$ with the topology of point-wise convergence.*

Proof. The representation of an arbitrary function $\varphi \in \mathcal{N}$ is by Nevanlinna's formula [1, p.92]

$$\varphi(z) = Az + B + \int \frac{1+tz}{t-z}d\gamma(t).$$

where A, B are real numbers, $A \geq 0$. We have the following inequality in U :

$$\frac{|z-\alpha_k|}{|z-t|} \leq \frac{1}{\sin\beta},$$

where β is the angle between the vector $z-\alpha_k$ and the real line. Lebesgue's convergence theorem applies to the function $\frac{1+tz}{t-z}(z-\alpha_k)$. I.e., since

$$\frac{1+tz}{t-z}(z-\alpha_k) \overset{z\to\alpha_k}{\longrightarrow} \begin{cases} 0 & if\ t \neq \alpha_k \\ -(1+\alpha_k^2) & if\ t = \alpha_k \end{cases}$$

we have

$$\lim_{z\to\alpha_k}(z-\alpha_k)\varphi(z) = -(1+\alpha_k^2)\gamma\left(\{\alpha_k\}\right).$$

Take $\epsilon_n, \epsilon > 0$, $\epsilon_n \to \epsilon$. Let $\{\delta_n\}$ be a sequence of atomic measures concentrated at the finite points in $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\delta_n \overset{*}{\to} 0$ and let

$$\varphi_n(z) = \varphi(z) + \epsilon_n z + \int \frac{1+tz}{t-z} d\delta_n(t).$$

Then $\varphi_n(z)$ satisfies the following properties:

$i)$ $(z-\alpha_k)\varphi_n(z) \overset{z\to\alpha_k}{\longrightarrow} -(1+\alpha_k^2)(\gamma+\delta_n)(\{\alpha_i\}) < 0$,
$ii)$ $z^{-1}\varphi_n(z) \overset{z\to\infty}{\longrightarrow} A + \epsilon > 0$.

Thus $\varphi_n \in \mathcal{N}_n$. Moreover, by the definition of φ_n, $\varphi_n \overset{n}{\to} \varphi$ point-wise, which proves our claim.

For $\varphi_n \in \mathcal{N}_n$, since $q_n(\alpha_k, \tau_n) \neq 0$ for $\alpha_k \neq 0$ and $\deg q_n(z, \tau_n) = n$ for $\alpha_k = \infty$, (44)-(45) equal

$$\begin{aligned} (z-\alpha_k)^{\alpha_k^\#} F(z) \quad if\ \alpha_k &\neq \infty, \\ z^{-\alpha_k^\#} G(z) \quad if\ \alpha_k &= \infty, \end{aligned}$$

where $F(z)$ is analytic about α_k and $F(z) = O(z-\alpha_k)$ when $z \to \alpha_k$. Analogously $G(z)$ is analytic about ∞ and $G(z) = O\left(\frac{1}{z}\right)$ when $z \to \infty$. Hence (39) holds for $\alpha \in \{\alpha_1, \alpha_1 \ldots, \alpha_{n-1}, \alpha_{n-1}\}$, $\mu = \mu_n$ and $\nu = \mu_n^\tau$. It remains to observe that $\Omega_{\mu_n}(i) - \Omega_{\mu_n^\tau}(i) = 0$. This is also true because μ_n is a probability measure (see argument that led to (36)), giving $\Omega_{\mu_n}(i) = i$ and $M(i) = \Omega_{\mu_n^\tau}(i) = i$ by the quadrature formula together with the normalization $M(1) = 1$. We have proved that for the given choice of φ_n, μ_n is a solution of the truncated moment problem in $\mathcal{R}_{n-1}$.

Following the proof of Th.11.8.1 in [5], since μ_n are probability measures one can extract a subsequence μ_{n_k} that converges weakly to a measure μ, solution of the moment problem in $\mathcal{L}_\infty$. The kernel $\frac{1+tz}{t-z}$ is continuous on $\mathbb{R}$ for $z \in U$. The weak convergence of the sequence μ_{n_k} implies

$$\zeta_{n_k}(z) = \int \frac{1+tz}{t-z} d\mu_{n_k}(t) \overset{k}{\to} \int \frac{1+tz}{t-z} d\mu(t).$$

On the other hand, by Proposition 12 c) and the point-wise convergence of φ_n to φ,

$$\zeta_{n_k}(z) = -\frac{A_{n_k}(z,x_0)\varphi_{n_k}(z) - C_{n_k}(z,x_0)}{B_{n_k}(z,x_0)\varphi_{n_k}(z) - D_{n_k}(z,x_0)} \overset{k}{\to} -\frac{A(z,x_0)\varphi(z) - C(z,x_0)}{B(z,x_0)\varphi(z) - D(z,x_0)}$$

so that

$$\int \frac{1+tz}{t-z} d\mu(t) = -\frac{A(z,x_0)\varphi(z) - C(z,x_0)}{B(z,x_0)\varphi(z) - D(z,x_0)}.$$

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