# Global upscaling of permeability in heterogeneous reservoirs; The output least squares (OLS) method.

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**Abstract**

This paper presents a new technique for computing the effective permeability on a coarse scale. It is assumed that the permeability is given at a fine scale and that it is necessary to reduce the number of blocks in the reservoir model. Traditional upscaling methods depend on local boundary conditions. It is well known that this approach often leads to ill-posed problems. We propose to compute a coarse scale permeability field that minimise the error, measured in a global norm, in the velocity and pressure fields. This leads to a well posed problem for a large number of reservoirs. We present several algorithms for finding the effective permeability values. It turns out that these algorithms are not significantly more computational expensive than traditional local methods. Finally, the method is illustrated by several numerical experiments.

## 1 Introduction

The last two decades it has become usual to model the permeability in a reservoir at a very fine scale, a scale much finer than it is possible to use in a reservoir simulator. The permeability is modelled at this scale because permeability is usually measured at a 0.1 -1 meter scale and it is easier to describe the geology at this scale. In order to bound the number of blocks in the reservoir simulator the block dimension is typically 10-100m vertically and 1-5 m horizontally. The fine scale description of the reservoir is usually generated by stochastic modelling, see [5]. It is then necessary to find upscaled permeability values for each coarse grid block.

Traditional upscaling methods compute block effective values of the permeability based on local boundary conditions, cf. e.g. [16] and [6]. If such effective properties exist[1], these methods give good estimates. In some cases

[1] The effective permeability exists if it is possible to define a value of the permeability on the coarse scale that gives the same flow as a fine scale simulation for all sets of boundary conditions.

an effective value does not exist, and the computations may lead to erroneous or arbitrary values, see [2].

In this paper a different approach is studied. The upscaling problem is formulated as a minimisation problem. More precisely, the upscaled permeability is defined as the permeability that minimises the difference between the pressure and the velocity fields generated by the fine and coarse scale pressure equations, respectively. Several norms, for measuring this difference, is discussed. By applying this approach, it is not necessary to make any assumptions on local boundary conditions (which in some cases will introduce ill-posed problems). Upscaling techniques based on minimising global norms are usually well-posed and leads to stable algorithms. However, this approach introduces some new technical problems; The upscaled permeability values may depend on the solution of the fine scale pressure equation, and minimisation problems are usually harder to solve than the linear equation systems arising in local upscaling methods.

Three different norms for comparing the difference in the pressure and velocity for a fine and coarse scale field is discussed. If the energy norm or inverse energy norm (the WOLS scheme, see [12]) is applied, it turns out that it is not necessary to solve the fine scale pressure equation in order to determine the coarse scale permeabilities. Promising results for the WOLS method are reported in [12]. In particular, this technique provides accurate representations of thin barriers. However, for some reservoirs these two methods fail to "preserve" the flow pattern on the coarse scale, cf. section 7. In fact, in some cases these techniques have many of the same properties as methods based on arithmetic and harmonic averaging of the permeability field.

Assume that the $L_2$ norm is used to measure the difference between the fine and coarse scale velocity and pressure fields. Then it is necessary to compute the fine scale pressure in order to solve the associated minimisation problem. This leads to a large linear equation system where the number of unknowns is the number of fine scale blocks. However, the most efficient linear equation solvers today use computer time proportional to the number of unknowns, see e.g. [9] or [4]. Recall that if local upscaling techniques are applied then a fine scale problem is solved on each coarse block. Thus, the computing time needed for solving one fine scale pressure equation, defined on the entire reservoir, is comparable to the time needed by local upscaling methods for solving all of the fine scale problems on the coarse blocks. Furthermore, it turns out that the associated minimisation problem can be solved very efficiently, see section 6. Hence, we conclude that upscaling techniques based on minimising the error in the global $L_2$ norms are not significantly more computational expensive than local methods.

The $L_2$ norm approach to upscaling seems to have promising properties. More precisely, in most coarse grid blocks the total mass flux over each coarse grid block interface is preserved. Moreover, we obtain very accurate

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flow rates in the wells. Finally, the pressure in most coarse blocks is simply the average pressure in the fine scale blocks inside the coarse grid block. In section 8 we present several numerical experiments illustrating the properties of this new upscaling method.

We introduce three algorithms for computing the transmissibilities of the coarse grid block interfaces that solves the $L_2$ minimisation problem. Transmissibilities give better resolution than permeabilities, and some problems related to non-rectangular blocks are avoided. If an effective permeability exists, the $L_2$ minimisation problem gives the same transmissibilities as traditional local upscaling methods.

In applications of our method, it is necessary to solve the fine scale pressure equation each time the rates or pressure in the reservoir changes considerably. This pressure field is used to compute the upscaled transmissibilities for the coarse grid blocks. These transmissibilities are applied in the following time steps in the reservoir simulator. In the time steps where the fine scale permeability solution is computed, the coarse scale pressure and velocity fields give exactly the same total flow rates as the fine scale solution. If the reservoir performance is stable, the upscaled permeabilities should also give accurate flow patterns in the following time steps, provided that the main flow directions are approximately unchanged.

The method is expected to be applicable for reservoirs where traditional local upscaling techniques fail. This is the case in reservoirs with high contrasts and barriers or high permeable zones at the size of the coarse scale grid blocks. It has also been observed that local upscaling methods quite often fail to handle the well blocks properly. In such cases, global methods seems to provide an interesting and accurate alternative, see section 8.

If an effective non-diagonal permeability tensor exists, transmissibility cross terms should be used. We described an algorithm for calculating these terms, cf. section 6.3. In this paper we focus on single phase, incompressible flow. Like most upscaling techniques it may also be generalised to multiphase, compressible flow. This issue is discussed in section 6.4.

## 2 Local methods

Traditional upscaling methods of permeability make assumptions on local boundary conditions, see e.g [16] and [6]. These methods work well if the variation of the saturations and velocities are limited on the fine grid block scale. If this is not the case, local methods may fail. In general, the error in the estimates are large if the assumed boundary conditions are far from the actual boundary conditions in the reservoir. Use of skin and non-diagonal tensors may reduce this effect, but the problem is still significant.

Effective properties are only well-defined if the separation of scale condition is satisfied, see [7];

$$l_k, l_S \ll L \ll L_k, L_S, L_p$$

where $l_k, l_S, L, L_k, L_S, L_p$ are the length scales of the small scale variation in the media, of the small scale variation in the saturations, the grid dimension, of the large scale variation in the media, of the large scale variation in the saturation and pressure, respectively. If this condition is not satisfied, the flow properties depend on the boundary conditions. Hence upscaling methods based on one set of boundary condition can not be expected to work well under other boundary conditions. This is the case if there are large changes in the pressure gradients (for example close to the wells), or large contrasts in the reservoir properties at the scale of the grid blocks. Figure 1 shows some typical blocks where local upscaling methods may fail to define an acceptable upscaled permeability field.

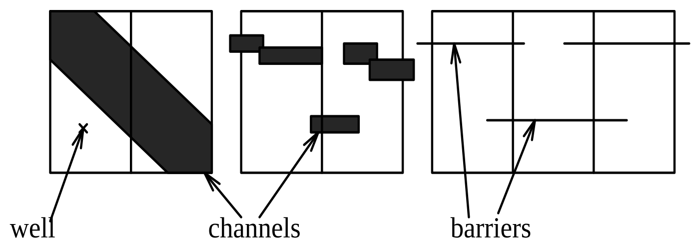


Figure 1: The figure shows some typical blocks where local upscaling methods may fail to define an acceptable upscaled permeability field.

## 3 Global norms

Assume that a fine scale description of a reservoir $k_h$ (permeability) is available and that this description has too many grid blocks to be used in a reservoir simulator. The equation

$$\mathrm{div}(k_h \nabla p_h) = 0 \tag{1}$$

with a certain set of reservoir boundary conditions gives the steady state pressure $p_h$ and velocity $v_h = -k_h \nabla p_h$. In this paper a set of reservoir boundary conditions is defined as a specification of the pressure or rates in the wells and a specification of the fluxes (in or out) of the reservoir. The challenge is to find a coarse scale description of the permeability $k_H$ such that the corresponding coarse scale pressure $p_H$, defined by

$$\mathrm{div}(k_H \nabla p_H) = 0 \tag{2}$$

with the same reservoir boundary conditions as in (1), and velocity $v_H = -k_H \nabla p_H$, are good approximations of $p_h$ and $v_h$, respectively. Hence, we want to find $k_H$ that minimises $\|p_h - p_H(k_H)\|$ and $\|v_h - v_H(k_H)\|$ (we write $p_H(k_H)$ and $v_H(k_H)$ to emphasize that the coarse pressure and coarse velocity fields are functions of $k_H$). In general, we may expect $k_H$ to depend on both the fine scale pressure $p_h$ and on the fine scale velocity $v_h$.

The steady state reservoir boundary conditions observed at a particular time step will usually not be satisfied at later time steps. The rates in the wells and the saturation distribution change with time in multiphase reservoirs. Hence, $k_H$ becomes time dependent. However, if the reservoir model is well-posed, small changes in the boundary condition will only lead to small changes in the coarse scale permeabilities. It seems reasonable to only update $k_H$ if large changes in the rates or large changes in the saturation has been observed. This issue is discussed in section 6.4.

Clearly, we can use different norms to measure the errors $\|p_h - p_H\|$ and $\|v_h - v_H\|$, introduced in the pressure field and in the velocity field by the upscaling process. The coarse scale permeabilities we compute will depend heavily on the norm we choose. In this paper three different norms are discussed: the energy norm, the inverse energy norm (the WOLS-scheme) and the $L_2$ norm. But, for reasons presented in section 7, we will emphasize on the $L_2$ norm.

# 4 Energy norm

In theoretical studies of the pressure equation, or more general, in studies of second order elliptic differential equations, mathematical properties are frequently stated in terms of the so-called energy norm, cf. e.g. [10] and [3]. If we apply this norm, our minimisation problem takes the form; Find $k_H$ and $p_H = p_H(k_H)$ such that (2) is satisfied and

$$\|p_h - p_H\|_E^2 = \int_\Omega \nabla(p_h - p_H) k_h \nabla(p_h - p_H)\, dx$$

is as small as possible. Here, $\Omega$ represents the domain of the reservoir in question. The finite element method, see [15] or [3], minimises the error in this energy norm. Applying the finite element method, defined on the coarse grid, to the pressure equation leads to a linear equation system where the discretized coarse pressure $p_H$ is the unknown. The coefficients in this linear equation system are given by integrals on the form

$$\int_\Omega \nabla\phi_i k_h \nabla\phi_j\, dx,$$

where $\{\phi_i\}$ are test functions with local support. Hence, the solution $k_H$ of the energy norm minimisation problem may be found by simple integrals of $k_h$ multiplied by the test functions. Hence, it is easy to find the coarse permeabilities in this case, but the estimates are close to the arithmetic mean of the permeability. This estimate does not honour thin barriers sufficiently and will overestimate the flow, see section 7.

## 5 Inverse energy norm (the WOLS scheme)

This method was introduced by Nielsen and Tveito in [12], and is defined as follows; Find $k_H$ and $p_H = p_H(k_H)$ such that (2) is satisfied and

$$\|v_h - v_H\|^2_{E^{-1}} = \int_\Omega (v_h - v_H) k_h^{-1} (v_h - v_H)\, dx \tag{3}$$

is as small as possible. Since the inverse $k_h^{-1}$ of the fine scale permeability is used as a weight function the method is referred to as the weighted output least squares (WOLS) method. As for the energy norm, discussed above, it turns out that it is not necessary to solve the fine scale pressure equation (1) in order to minimise the functional given in (3). However, it is necessary to solve a minimisation problem defined on the coarse scale. This makes the method computationally more expensive than the energy norm minimisation algorithm. If the number of fine scale blocks is considerably larger than the number of coarse grid blocks, then this method will be considerably faster than local upscaling methods. Some promising results are presented in [12]. In particular, the scheme provides accurate representations on the coarse scale of low permeable zones acting as barriers. However, as illustrated in section 7, low permeability blocks that do not function as barriers may have too large effect on the estimates. The estimates are in some cases close to the harmonic mean of the fine scale permeability and may underestimate the flow.

## 6 $L_2$ norm (the OLS scheme)

Clearly, on a coarse grid it is not possible to reconstruct the fine scale pressure $p_h$ exactly. For each coarse grid block $i$, let $p_r^i$ denote the average of the corresponding fine scale pressure $p_h$, and for each coarse grid block side $i,j$, let $q_r^{i,j}$ denote the sum of the corresponding fine scale mass flux, i.e. $q_r^{i,j}$ represents the total mass flux across the interface $i,j$. A reasonable ambition on the coarse grid for an accurate upscaling method should be that the pressure in each coarse block equals $p_r$ and that the flux over each coarse grid block interfaces equals $q_r$.

It turns out that minimising the $L_2$ norms

$$\|p_h - p_H\|^2_{L_2} = \int_\Omega (p_h - p_H)^2\, dx \tag{4}$$

and

$$\|v_h - v_H\|^2_{L_2} = \int_\Gamma ((v_h - v_H)\cdot n)^2\, dx \tag{5}$$

give solutions that are close to $p_r = \{p_r^i\}_i$ and $q_r = \{q_r^{i,j}\}_{i,j}$, respectively[2]. Here, $\Gamma$ represents the union of all the boundaries of the coarse scale grid

[2] The fluxes $\{q_r^{i,j}\}_{i,j}$ are uniquely determined by the velocities $\{v_H^{i,j}\}_{i,j}$ at the interfaces and the grid parameters, see equation (9).

blocks, and $n$ is the outwards directed normal vector (of unit length) to these boundaries.

In a finite difference reservoir simulator the permeabilities are only used for calculating the transmissibilities[3] $T_H$, which in turn are used in the flow calculations. It is partly unstable to compute $k_H$ from the transmissibilities. Typically, these instabilities lead to a "chess board pattern" for $k_H$ instead of decreasing the error. In addition, transmissibilities give better resolution than permeabilities. Moreover, by computing the transmissibilities directly, it is easier to handle irregular grids. Thus, we will focus on computing transmissibilities instead of permeabilities in the algorithms presented below.

In order to reproduce $p_r^i$ and $q_r^{i,j}$ on the coarse grid, the transmissibilities must satisfy the equation

$$T_H^{i,j}(p_r^i - p_r^j) = q_r^{i,j}, \tag{6}$$

where $i$ and $j$ are neighbour blocks and $T_H^{i,j}$ represents the transmissibility of the associated block interface. The reason for this is; The discretized pressure equation (2) can be written on the form

$$\sum_j (p_H^i - p_H^j) T_H^{i,j} = b_H^i \tag{7}$$

where the right hand side $b_H^i = 0$, except in blocks containing a well(s) or blocks at the border of the reservoir (where there might be an in or out flux of mass). Equation (7) expresses mass conservation since the mass flux in and out of block $i$ adds up to zero. Clearly, since the mass is conserved on the fine scale and $q_r^{i,j}$ is the total mass flux across the interface, it follows that if $T_H^{i,j}$ satisfies (6), for all $i, j$, then (7) must hold. In fact, it is sufficient that the right hand side $b_H^i$ satisfies

$$b_H^i = \sum_j q_r^{i,j} \tag{8}$$

for all $i$. That is, if (6) and (8) hold then $p_r^i$ and $q_r^{i,j}$ are reproduced on the coarse grid.

The mass flux from block $i$ to block $j$ is

$$q_H^{i,j} = A_{i,j} v_H^{i,j} \cdot n_H^{i,j} = T_H^{i,j}(p_H^i - p_H^j) \tag{9}$$

where $A_{i,j}$ represents the area of the block interface between blocks $i$ and $j$. Note that there is a linear relationship between the mass flux $q$ and the velocity $v \cdot n$. Hence, minimizing the $L_2$ norm, with respect to $\{T_H^{i,j}\}$, in either the fluxes or velocities gives the same solution.

[3] The transmissibility of an interface, in a regular grid, is defined as the harmonic average of the permeability in the two blocks multiplied by the area of the common interface and divided by the distance between the two grid block centers.

Even in cases where the fine scale permeability $k_h$ is positive and bounded, the solution $T_H$ that minimises the two norms (5) and (4) may be negative or infinite large, see [2]. In applied problems, it seems to be necessary to find $T_H \in [T_L, T_U]$ that minimises the two $L_2$ norms. A possible choice of bounds for the transmissibilities could be the local arithmetic/harmonic and harmonic/arithmetic means of the permeabilities in the two blocks. If an effective value exists, it will always belong to this interval. It turns out that it is necessary to use $L_2$ norms in both pressure and velocity in order to get a unique solution.

For a given set of admissible transmissibilities $T_H = \{T_H^{i,j}\}_{i,j}$, let $p_H = p_H(T_H)$ denote the solution of the coarse scale pressure equation (7) and let $v_H = v_H(T_H)$ represent the associated velocity field. Then the $L_2$ minimisation problem can be formulated as follows;

**$\mathbf{L}_2$ minimisation problem**

*For each pair $i, j$ of neighbouring coarse grid blocks find $T_H^{i,j} \in [T_L^{i,j}, T_U^{i,j}]$ such that $\|v_h - v_H(T_H)\|_{L_2}$ is minimised, and of all solutions that minimise this functional, minimise $\|p_h - p_H(T_H)\|_{L_2}$. In case both functionals are invariant for a transmissibility $T_H^{i,j}$, set $T_H^{i,j} = (T_L^{i,j} T_U^{i,j})^{1/2}$.*

It is trivial to show that the functions $p_r$ and $q_r$, defined at the beginning of this section, minimise[4] the two $L_2$ norms. However, the corresponding transmissibility defined by

$$T_1^{i,j} = q_r^{i,j}/(p_r^i - p_r^j),$$

cf. equation (6), where the flow $q_r^{i,j}$ is from block $i$ to block $j$, need not satisfy

$$T_L^{i,j} \leq T_1^{i,j} \leq T_U^{i,j}. \tag{10}$$

If (10) is satisfied at all block interfaces, then $T_H^{i,j} = T_1^{i,j}$ is the solution of the minimisation problem. If $p_r^i = p_r^j$ and $q_r^{i,j} = 0$, then the two norms are invariant with respect to $T_H^{i,j}$, and the $L_2$ minimisation solution is $T_H^{i,j} = (T_L^{i,j} T_U^{i,j})^{1/2}$.

It is believed that for most reservoirs, and for reasonable bounds $T_L^{i,j}$ and $T_U^{i,j}$, inequality (10) will be satisfied at almost all block interfaces in the reservoir. The theorem and the algorithms presented in this paper are based on the principle that the pressure solution $p_H$ of the minimisation problem is equal to $p_r$ in all blocks, except in a limited neighbourhood to block interfaces where (10) is not satisfied, see section 6.2. Moreover, the mass flux $q_H$ will be equal to $q_r$ except in parts of some of these neighbourhoods. We prove that for a certain set of reservoirs the solution of the minimisation problem satisfies this local property.

[4]Recall that we can apply (9) to compute $v_H$ if $q_H$ is given, and vice versa.

In the literature addressing the theory of so called inverse problems, the solution concept discussed above is frequently applied to solve parameter identification problems, cf. e.g. [1] or [8]. The approach is referred to as the output least squares (OLS) method. Therefore, we will refer to the new upscaling scheme as the OLS method.

## 6.1 Algorithms

The algorithms are based on first finding the global fine scale pressure $p_h$. It seems impossible to compute the solution of the $L_2$ minimisation problem without finding $p_h$ first. In local methods, as described in e.g. [16], it is also necessary to compute local approximations of $p_h$ in order to determine the upscaled permeability field. However, in the latter case, this is done for each coarse block separately assuming a local boundary condition. The most important advantage of the energy norm approach and WOLS scheme, described in sections 4 and 5, is that this step is avoided.

In this section we present two algorithms for computing the solution of the $L_2$ minimisation problem presented in the previous section. Algorithm 1 is easy to implement, and determines the exact solution of the problem for a certain class of reservoirs called $A_1$. In this context a reservoir is defined by a fine scale permeability field $k_h$, a set of coarse blocks and a set of boundary conditions for the fine scale problem. Outside the set $A_1$, the algorithm gives an approximate solution. Algorithm 2 is more complex to implement. However, it gives the exact solution for a larger set of reservoirs $A_2 \supseteq A_1$. Outside this set, it gives an approximate solution.

**Algorithm 1**

1. Find the solution $p_h$ of the fine scale pressure equation and compute the associated Darcy velocity $v_h$.
2. Compute the $L_2$-projections $p_r$ and $q_r$ of $p_h$ and $q_h$ onto the coarse grid.
3. For all pairs of neighbour blocks $i, j$:
   (a) $T_1^{i,j} := q_r^{i,j}/(p_r^i - p_r^j)$,
   where $q^{i,j}$ represents the mass flux from block $i$ to block $j$.
   (b) Set $T_H^{i,j} \in [T_L^{i,j}, T_U^{i,j}]$ such that $|T_H^{i,j} - T_1^{i,j}|$ is minimised.

Clearly, algorithm 1 finds the unique minimum of the $L_2$ minimisation problem if (10) is satisfied for all pairs of neighbours. The set $A_1$ is the set of reservoirs where this condition is satisfied.

The main purpose of Algorithm 2 is to try to minimise the number of block interfaces at which condition (10) is not satisfied. The basic idea behind the algorithm is to make small, and local, changes in the projected pressure field and update the transmissibilities in an iterative procedure.

More precisely, assume that $T_1^{i,j} > T_U^{i,j}$. Then we redefine the transmissibility at this interface by setting

$$T_2^{i,j} = T_U^{i,j}.$$

In addition we assign new pressure values $\tilde{p}_r^i$ and $\tilde{p}_r^j$ to blocks $i$ and $j$ by requiring that Darcy's law is fulfilled

$$(\tilde{p}_r^i - \tilde{p}_r^j)T_2^{i,j} = q_r^{i,j},$$

and by assuring that the change in the pressure field is as small as possible, i.e. by minimising the following functional with respect to $\tilde{p}_r^i$ and $\tilde{p}_r^j$

$$(\tilde{p}_r^i - p_r^i)^2 + (\tilde{p}_r^j - p_r^j)^2.$$

The case $T_1^{i,j} < T_L^{i,j}$ is handled analogously.

Clearly, if the pressure field is changed in this way, then condition (10) holds at the interface between blocks $i$ and $j$. However, this approach may introduce new pressure differences (drops) in the neighbourhood of blocks $i$ and $j$ such that (10) is violated. Hence, the method must be applied in an iterative fashion leading to the following algorithm.

**Algorithm 2**

1. Find the solution $p_h$ of the fine scale pressure equation and compute the associated Darcy velocity $v_h$.

2. Compute the $L_2$-projections $p_r$ and $q_r$ of $p_h$ and $q_h$ onto the coarse grid. Set $\tilde{p}_r = p_r$.

3. While not convergence do

   (a) For all pairs of neighbour blocks $i, j$:

      i. $\tilde{T} := q_r^{i,j}/(\tilde{p}_r^i - \tilde{p}_r^j)$,
      where $q^{i,j}$ represents the mass flux from block $i$ to block $j$.

      ii. Set $T_2^{i,j} \in [T_L^{i,j}, T_U^{i,j}]$ such that $|T_2^{i,j} - \tilde{T}|$ is minimised.

      iii. If $\tilde{T} \notin [T_L^{i,j}, T_U^{i,j}]$ set

$$\begin{aligned} \tilde{p}_r^i &:= 0.5 * \left( \tilde{p}_r^i + \frac{q^{i,j}}{T_2^{i,j}} + \tilde{p}_r^j \right), \\ \tilde{p}_r^j &:= \tilde{p}_r^i - \frac{q^{i,j}}{T_2^{i,j}}. \end{aligned}$$

4. For all pairs of neighbour blocks $i, j$:

- Set $T_H^{i,j} \in [T_L^{i,j}, T_U^{i,j}]$ such that $|T_H^{i,j} - T_2^{i,j}|$ is minimised.

In this paper we apply the following stopping criteria in the algorithm (cf. step 3. in Algorithm 2); Let $q(it)$ represent the number of interfaces, in iteration number $it$, at which condition (10) is not satisfied. The iteration process is terminated if the average relative reduction in $q(it)$, in the last 10 iterations, is less than 1%.

The number of interfaces at which condition (10) is violated can be reduced significantly by applying algorithm 2. However, $q(it)$ will in most realistic cases not reach 0 within an acceptable number of iterations. Hence, step 4. in the algorithm is needed to assure that all the transmissibilities satisfy their predefined bounds. The set $A_2$, introduced above, is simply defined to be the set of reservoirs for which $q(it)$ reaches 0.

Finally, comparing algorithms 1 and 2, it turns out that the representation of the flow pattern, and consequently the production/injection rates in the wells, on the coarse scale is improved significantly by applying Algorithm 2 instead of Algorithm 1, see section 8.

It should also be mentioned that, throughout this paper, we will refer to both algorithms 1 and 2 as the OLS scheme.

### 6.2 Properties

In this section it is proved some theoretical properties of the $L_2$ minimisation problem and the algorithms presented in the previous section. The theoretical properties are proved by introducing Algorithm 3 defined below. The algorithm is not intended for implementation, it is only a constructive method to prove the properties of the problem.

**Theorem**

*For a certain class of reservoirs $A_3$ (fine scale permeability $k_h$, a set of boundary conditions for the reservoir and coarse grid blocks), there exists a unique solution of the $L_2$ minimisation problem. The solution of the $L_2$ minimisation problem satisfies $p_H^i(T_H) = p_r^i$ in all coarse grid blocks, and $q_H^{i,j}(T_H) = q_r^{i,j}$ on all coarse grid block interfaces except in local neighbourhoods around coarse blocks boundaries where (10) is not satisfied. These neighbourhoods are characterised by $p_H^i(T_H) \neq p_r^i$. In some of the local neighbourhoods $q_H^{i,j}(T_H) = q_r^{i,j}$ is satisfied on all block boundaries while other local neighbourhoods have $q_H^{i,j}(T_H) \neq q_r^{i,j}$ on some of the grid block boundaries.*

*Algorithm 1, 2 and 3 give the exact solution of the $L_2$ minimisation problem for reservoirs in respectively $A_1 \subset A_2$ and $A_1 \subset A_3$. The set $A_1$ is described by the reservoir where (10) is satisfied on all coarse grid block boundaries. The set $A_2$ is described by the reservoir where $q(it)$ reaches 0. The set $A_3$ is described as the set of reservoirs where it is possible to remove* *the statement " AND $n(D \cup \{k\}) < 3$ AND $k$ not member of a set $D$ previous in the algorithm " from the algorithm 3 below, without changing the performance of the algorithm.*

In Algorithm 3 it is necessary with some mathematical notation. $D$ is used to denote a subset of the set of coarse grid blocks. The blocks in the set $D$ changes during the algorithm. The algorithm needs the following three functions of the set $D$: $n(D)$, the number of parallel path in $D$, $p^i(D)$ the pressure in block $i \in D$ and $T^{i,j}(D)$ the transmissibility between block $i$ and $j$ where $i, j \in D$. $p^i(D)$ and $T^{i,j}(D)$ are defined as the solution of the $L_2$ minimisation problem restricted to $D$. This minimisation is performed without assuming any boundary condition at the border of $D$. The set $D$ is increased until either

$$T^{i,k}(D)(p^i(D) - p_r^k) = q_r^{i,k} \tag{11}$$

for all $i \in D$ and $k \notin D$ is satisfied at the border of $D$ or $D$ is so large that the algorithm only finds an approximate solution of the $L_2$ minimisation problem. It is described how these function are computed after the algorithm.

**Algorithm 3**

1. Find the solution $p_h$ of the fine scale pressure equation and compute the associated Darcy velocity $v_h$.
2. Compute the $L_2$-projections $p_r$ and $q_r$ of $p_h$ and $q_h$ onto the coarse grid.
3. For all neighbours $i, j$ not evaluated earlier:
   (a) $T_1^{i,j} := q_r^{i,j}/(p_r^i - p_r^j)$
   where $q^{i,j}$ represents the mass flux from block $i$ to block $j$.
   (b) If $T_L^{i,j} \leq T_1^{i,j} \leq T_U^{i,j}$, set $T_H^{i,j} := T_1^{i,j}$
   (c) else
      i. $D := \{i, j\}$
      ii. Do while ( $\exists k \notin D, i \in D$, where $i$ and $k$ are neighbours, AND $q_r^{i,k} \notin [(p^i(D) - p_r^k)T_L^{i,k}, (p^i(D) - p_r^k)T_U^{i,k}]$ AND $n(D \cup \{k\}) < 3$ AND $k$ not member of a set $D$ previous in the algorithm ) set $D := D \cup \{k\}$
      iii. $T_H^{i,j} := T^{i,j}(D)$

Note that in step 3c(iii) the algorithm may overwrite $T_H^{i,j}$ that is set in step 3b earlier, but step 3b will not overwrite any value of $T_H^{i,j}$ that is set earlier.

A flow path is defined as a sequence of blocks where there is mass flux from one block in the sequence to its successor in the sequence. The flow path in $D$ may branch or there may be parallel pathes, see figure 2. Parallel pathes are defined as two path that both run through two blocks $i$ and $j$ and that the two path are different between $i$ and $j$. The number of parallel pathes in $D$, $n(D)$ is defined as the largest number of different path that run through two blocks $i, j \in D$ and are different between $i$ and $j$.

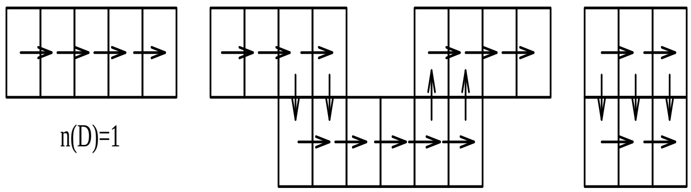


Figure 2: The number of parallel path $n(D)$ for different configurations $D$. The arrows gives the flux direction.

The following three paragraphes describe how to calculate the functions $p^i(D)$ and $T^{i,j}(D)$ as defined above. Assume that $q_r^{i,j} < (p_r^i - p_r^j)T_L^{i,j}$ and $q_r^{i,j}(p_r^i - p_r^j) > 0$ for the first two blocks in $D$. This is set in statement 3c(i) in the algorithm. The case $q_r^{i,j} > (p_r^i - p_r^j)T_U^{i,j}$ or $q_r^{i,j}(p_r^i - p_r^j) \leq 0$ is treated correspondingly but then $T_L$ is replaced by the upper limit $T_U$. The problem in $D$ is that the pressure drop between $p_r^i$ at the influx and outflux boundary of $D$ is too large compared with the mass flux. The challenge is to find $T^{i,j}(D)$ such that the flow is as small as possible. Assume first that $n(D) = 1$. In this case it is possible to obtain the mass flux $q_r$ over all block sides in the coarse model i.e.

$$(p^i(D) - p^j(D))T^{i,j}(D) = q_r^{i,j} \tag{12}$$

and it is left to minimise $\|p(D) - p_h\|_D$, the norm (4) integrated over $D$. It is easy to show that $T^{i,j}(D) = T_L^{i,j}$. The difference in the pressure between two blocks is easily calculated from $q_r^{i,j}/T_L^{i,j}$ at all block interfaces in the path between the two blocks. Then the norm $\|p(D) - p_h\|_D$ is a quadratic function with one unknown pressure that is straightforward to minimise.

The case $n(D) = 2$ is a somewhat more complicated. Let $k_1$ and $k_2$ denote the blocks that is common in the two parallel pathes. If all transmissibilities in $D$ had been $T_L^{i,j}$, the two path would most likely specify different pressure drop between $k_1$ and $k_2$. The path that specify the smallest pressure drop between $k_1$ and $k_2$ is the critical one, since the problem in $D$ is that the pressure drop is too large. It is easily shown that $T^{i,j}(D) = T_L^{i,j}$ in the critical path. Assume first that it is possible to achieve this pressure drop in the non-critical path between $k_1$ and $k_2$ by having $T_L^{i,j} \leq T^{i,j}(D) \leq T_U^{i,j}$.

Then it is possible to obtain (12) and it is left to minimise $\|p(D) - p_h\|_D$. Set $T^{i,j}(D)$ in the non-critical path such that one obtains the same pressure difference between the blocks $k_1$ and $k_2$ as in the critical path and such that $p(D)$ is either as small as possible or as large as possible. One of these two solutions will minimise $\|p(D) - p_h\|_D$. Then the difference in pressure between two blocks in $D$ is easily calculated from $q_r^{i,j}/T^{i,j}(D)$ on a path between the two blocks. The norm $\|p(D)-p_h\|_D$ becomes a quadratic function with one unknown that is easy to minimise.

If it is not possible to achieve the same pressure drop between $k_1$ and $k_2$ in the non-critical path for $T^{i,j} \leq T_U^{i,j}$, as in the critical path, it is not possible to obtain (12). Then it is necessary to move some of the mass flux from the critical path to the non-critical path such that we get the same pressure drop with $T^{i,j}(D) = T_L^{i,j}$ in the critical path and $T^{i,j}(D) = T_H^{i,j}$ in the non-critical path. This will minimise the deviation in the $\|(v(D) - v_h) \cdot n\|_D$ norm. Also in this case $\|p(D) - p_h\|_D$ is a quadratic function with one unknown that is easy to minimise.

The case with $n(D) > 2$ is more complex and the algorithm stops to increase $D$ before $n(D) > 2$. In these cases the algorithm only minimise the norms (5) and (4) in each set $D$ isolated, without satisfying (6) at the boundary. The algorithm is in this case only able to find an approximate solution to the $L_2$ minimisation problem.

**Proof**

It is trivial to see that the $L_2$ projections $p_r$ and $q_r$ minimise the two $L_2$ norms in the $L_2$ minimisation problem. This implies the part of the theorem regarding algorithms 1 and 2. It is left to prove the properties of the $L_2$ minimisation problem using algorithm 3.

Each set $D$ corresponds to the local neighbourhoods where $p_H^i \neq p_r^i$. The set $D$ is only introduced if (10) is not satisfied. Each time a domain $D$ is generated, the algorithm minimise first the functional (5) and then the functional (4) integrated over $D$. The set $D$ is increased until the boundary condition (11) is satisfied at the border of $D$. Then the solution in the entire reservoir consists of blocks where $p_H^i = p_r^i$ and $q_H^{i,j} = q_r^{i,j}$ on all block sides and possibly one or several domains $D$ where the transmissibilities minimise the two functionals. If $n(D) = 1$, then $q_H^{i,j} = q_r^{i,j}$ in $D$, and if $n(D) = 2$, then there may be grid block boundaries where $q_H^{i,j} \neq q_r^{i,j}$ in $D$. The algorithm clearly finds the solution of the $L_2$ minimisation problem as long as the statement " AND $n(D \cup \{k\}) < 3$ AND $k$ not member of a set $D$ previously defined in the algorithm" is not invoked in the algorithm.

## 6.3 Non-Diagonal permeability tensors

If the blocks are rectangular and there exists an effective diagonal permeability, algorithms 1-3 give the same values of the transmissibilities as a standard local upscaling method gives by first calculating effective permeabilities and then calculating the transmissibilities by harmonic averaging. Hence, they also give the exact solution if the reservoir boundary conditions are changed linearly. If there are large off-diagonal elements in the fine scale permeability, then the transmissibilities found by Algorithms 1-3 depend heavily on the boundary condition. This may be avoided by modelling the flow between blocks that do not have a common side, i.e. cross terms. Algorithm 4 described below, gives a finite difference approximation to the differential equation that is exact when there exists an effective permeability tensor and the pressure field is a linear function.

One set of boundary conditions uniquely defines diagonal transmissibilities, $T_H$. It is necessary with several different sets of boundary conditions in order to define transmissibilities $S_H$ where there may be cross terms. We have defined the $L_2$ minimisation problem with cross terms such that the flow is identical with the $L_2$ minimisation problem for one particular set of boundary conditions except that the flow from block $i$ to $j$ and then from $j$ to $k$ is allowed to flow directly from $i$ to $k$. The additional flexibility from the cross term $i, k$ is used to minimise the norm (5) integrated over block side $i, j$ and $j, k$ for the other sets of boundary conditions. The reason for picking out one particular set of boundary condition is to reduce the computing time. In an application where there are different sets of boundary in different time intervals (see section 6.4), this particular set of boundary condition may be the set of boundary condition that is believed to be most valid for this time interval, while the other sets of boundary conditions are the set of boundary condition for other time intervals. The minimisation problem with cross terms is formulated with mass fluxes instead of velocities in order to simplify the notation.

**$\mathbf{L_2}$ minimisation problem with cross terms** *Define the transmissibilities including cross terms $S_H^{i,k}$ by*

1. *Let $T_\alpha^{i,j}$ be the diagonal transmissibilities from the $L_2$ minimisation problem for the sets of boundary conditions $\alpha = 1, ..., m$, with corresponding pressure $p_\alpha^i = p_{H,\alpha}^i$ and fluxes $q_\alpha^{i,j} = T_\alpha^{i,j}(p_\alpha^i - p_\alpha^j)$ where $m > 1$.*

2. *Define which cross terms $i, k$ to include, requiring that $i$ and $j$ are neighbours and $j$ and $k$ are neighbours.*

3. *Determine* $S_H = \{S^{i,k}\}_{i,j}$ *such that for each cross term separately*

$$\sum_{\alpha=1}^{m-1}((q^{i,j}(S_H) - q_\alpha^{i,j})^2 + (q^{j,k}(S_H) - q_\alpha^{j,k})^2) \tag{13}$$

*is minimised under the constraint*

$$S^{i,k}(p_m^i - p_m^k) = (T_m^{i,j} - S^{i,j})(p_m^i - p_m^j) = (T_m^{j,k} - S^{j,k})(p_m^j - p_m^k). \tag{14}$$

*where*

$$q^{i,j}(S_H) = S^{i,j}(p_\alpha^i - p_\alpha^j) + S^{i,k}(p_\alpha^i - p_\alpha^k)$$
$$q^{j,k}(S_H) = S^{j,k}(p_\alpha^j - p_\alpha^k) + S^{i,k}(p_\alpha^i - p_\alpha^k).$$

The constraint (14) implies that for the set of boundary conditions corresponding to $\alpha = m$ is part of the flux from $i$ to $j$ and $j$ to $k$ substituted by a direct flux from $i$ to $k$. In (13) it is summed over the other boundary conditions evaluated. In the algorithm below it is included cross terms if the diagonal transmissibilities varies considerably between the different sets of boundary conditions. The following algorithm computes the cross terms:

**Algorithm 4**

1. Use another algorithm to find $T_\alpha^{i,j}$, the diagonal transmissibilities for the $L_2$ minimisation problem with corresponding pressure $p_\alpha = p_{H,\alpha}$ and flux $q_\alpha = q_{H,\alpha}$ for the set of boundary conditions $\alpha = 1, ..., m$ for $m > 1$.

2. Define cross terms in block $i$ if

$$\sum_\alpha \sum_j (T_\alpha^{i,j} - \bar{T}^{i,j})^2 > c$$

   where $\bar{T}^{i,j} = 1/m \sum_\alpha T_\alpha^{i,j}$ and $c$ is a constant. If $c < 0$ there are cross terms in all blocks. Define maximum 1(3) cross terms per block in 2D(3D). If neighbouring blocks have cross terms, let the cross terms be in the same direction.

3. Find the cross term $S^{i,k}$ that minimise (13) i.e.

$$S^{i,k} = \frac{\sum_{\alpha=1}^{m-1}((T_\alpha^{i,j} - T_m^{i,j})(p_\alpha^i - p_\alpha^j) + (T_\alpha^{j,k} - T_m^{j,k})(p_\alpha^j - p_\alpha^k))}{\sum_{\alpha=1}^{m-1}(2(p_\alpha^i - p_\alpha^k) - \frac{(p_n^i - p_n^k)(p_\alpha^i - p_\alpha^j)}{p_n^i - p_n^j} - \frac{(p_n^i - p_n^k)(p_\alpha^j - p_\alpha^k)}{p_n^j - p_n^k})}, \tag{15}$$

   and $S^{i,j}$ and $S^{j,k}$ from equation (14)

$$S^{i,j} = T^{i,j} - S^{i,k}\frac{p_m^i - p_m^k}{p_m^i - p_m^j}$$

and

$$S^{j,k} = T^{j,k} - S^{i,k}\frac{p^i_m - p^k_m}{p^j_m - p^k_m}$$

By applying (14) the functional (13) can be written on the following form

$$\sum_{\alpha=1}^{m-1}\Big(((T^{i,j}_\alpha - T^{i,j}_m)(p^i_\alpha - p^j_\alpha) + S^{i,k}(\tfrac{(p^i_m-p^k_m)(p^i_\alpha-p^j_\alpha)}{p^i_m-p^j_m} - p^i_\alpha + p^k_\alpha))^2 + \\ ((T^{j,k}_\alpha - T^{j,k}_m)(p^j_\alpha - p^k_\alpha) + S^{i,k}(\tfrac{(p^i_n-p^k_m)(p^j_\alpha-p^k_\alpha)}{p^j_m-p^k_m} - p^i_\alpha + p^k_\alpha))^2\Big).$$

The minimum of this expression is obtained for $S^{i,k}$ as defined in (15).

The cross terms are illustrated in two examples. The first example shows that the cross terms gives the exact solution for linear pressure fields when there is a constant full permeability tensor. Then it is necessary with one cross term in 2D and 3 cross terms in 3D. The second example is numerical.

**Example 1**
Let the effective full permeability tensor and velocity be

$$K = \begin{bmatrix} K_{xx}, & K_{xy}, & K_{xz} \\ K_{xy}, & K_{yy}, & K_{yz} \\ K_{zx}, & K_{zy}b, & K_{zz} \end{bmatrix}, \qquad v = \begin{bmatrix} K_{xx}a + K_{xy}b + K_{xz}c \\ K_{xy}a + K_{yy}b + K_{yz}c \\ K_{zx}a + K_{zy}b + K_{zz}c \end{bmatrix}.$$

for the linear pressure $p = -ax - by - cz$. The velocity is

$$v = \begin{bmatrix} S_x a + S_{xy}(a+b) + S_{xz}(a+c) \\ S_y b + S_{xy}(a+b) + S_{yz}(b+c) \\ S_z c + S_{xz}(a+c) + S_{yz}(b+c) \end{bmatrix}$$

for the numerical solution with cross terms. These two expressions are identical when $S_x = K_{xx} - K_{xy} - K_{xz}$, $S_y = K_{yy} - K_{xy} - K_{yz}$, $S_z = K_{zz} - K_{xz} - K_{yz}$, $S_{xy} = K_{xy}$, $S_{xz} = K_{xz}$, and $S_{zy} = K_{zy}$, which is the solution obtained by formula (15).

**Example 2** Assume that the effective permeability is

$$k_h = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

and that block no 2 is neighbour to blocks 1 and 3. In the example the cross term between block 1 and 3 is calculated. All block sides have length 1. Algorithm 4 gives the transmissibilities $S^{12} = S^{23} = S^{13} = 1$. The cross terms gives the correct velocity which may be calculated analytically. The example illustrates that when the off-diagonal terms are large, then cross terms are essential in order to model the flow correctly when the pressure gradient varies.

| $\alpha$ | p | $p^1_\alpha - p^2_\alpha$ | $T^{1,2}_\alpha$ | $p^2_\alpha - p^3_\alpha$ | $T^{2,3}_\alpha$ | $p^1_\alpha - p^3_\alpha$ |
|---|---|---|---|---|---|---|
| 1 | -2x-y | 1 | 4 | 2 | 2.5 | 3 |
| 2 | -2x+y | -1 | 0 | 2 | 1.5 | 1 |

Table 1: The transmissibilities and pressure differences for two different set of boundary conditions

| k | p | $S^{12} = 1, S^{23} = 1, S^{13} = 1$ | $T^{23}_k = 2.5, T^{12}_k = 4$ | $T^{23}_k = 1.5, T^{12}_k = 0$ |
|---|---|---|---|---|
| 1 | -2x-y | [5,4] | [5,4] | [3,0] |
| 2 | -2x+y | [3,0] | [5,-4] | [3,0] |
| 3 | -x | [2,1] | [2.5,0] | [1.5,0] |
| 4 | -x-y | [3,3] | [2.5,4] | [1.5,0] |

Table 2: The velocities for different transmissibilities with and without cross terms for different set of boundary conditions

### 6.4 Multiphase

Multiphase, compressible flow may be described by the equation

$$\operatorname{div}(a(S,p)\nabla p) = f(p, p_t, S, S_t) \tag{16}$$

where $S, S_t, p, p_t$ are saturations and pressure and their derivative with respect to time, see [14]. The mass flux between block $i$ and $j$ is

$$q^{i,j}_h = c_h(S) T^{i,j}_h (p^i_h - p^j_h). \tag{17}$$

where the relative permeabilities are the main contributor to $c_h$. When there has been a considerably change in the boundary conditions or the saturations, then it is necessary to update the transmissibilities. This may be done using the following algorithm where it is assumed that the pressure and saturation is known in each coarse grid block

**Algorithm 5**

1. Distribute the saturation in the fine grid blocks inside each coarse grid block using e.g. capillary equilibrium or viscous dominated flow.
2. Solve the pressure equation (16) at the fine grid assuming $a(S,p)$ and $f(p, p_t, S, S_t)$ are calculated from the coarse grid pressure and saturation distribution.
3. Find average pressure in each coarse block $p_r$, the mass flux over each coarse grid block side $q_r$ and the constant $c_h(S)$ in equation (17) from any method for upscaling relative permeabilities.
4. Find the transmissibilities $T^{i,j}_H \in [T^{i,j}_L, T^{i,j}_U]$ using algorithm 1 or 2 that minimise the $L_2$ minimisation problem using $p_r$ and $q_r$.

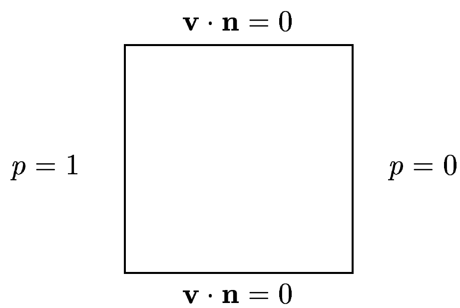


Figure 3: The solution domain/block $B$ and the first set of boundary conditions considered in section 7.

# 7 Analytical examples

The purpose of this section is to present some examples illustrating various properties of the global upscaling techniques discussed in sections 3-6. In this simple cases it is possible to solve the associated minimisation problems by hand and get explicit expressions for the upscaled permeability. These expressions are used to compare the various approaches to the problem. In section 8 we will present more realistic test problems along with their numerical solution.

For simplicity, we will consider a two dimensional reservoir in this section. Consider the fine scale pressure equation (1) subject to the boundary conditions given in Figure 3. Here, $B = (0, l) \times (0, h)$ represents the domain of an (academic) reservoir. We will assume that $B$ is composed of non-overlapping fine scale grid blocks $\{b_i\}_{i=1}^n$,

$$B = \bigcup_{i=1}^{n} b_i,$$

and that $k_h$ is a piecewise constant function, i.e.

$$k_h(x) = k_i \quad \text{for all } x \in b_i \text{ and } i = 1, \ldots, n,$$

where $\{k_i\}_{i=1}^n$ are positive constants.

Our goal is to determine an upscaled permeability value $k_H$ for $B$, i.e. we assume that the coarse grid consists of one block, namely $B$. Consider the coarse scale pressure equation (2), also subject to the boundary conditions given in Figure 3. Clearly, the solution $p_H$ of this problem is given by

$$p_H(x, y) = 1 - \frac{x}{l} \quad \text{for all } (x, y) \in B,$$

and is thus independent of $k_H$. However, the associated velocity field $v_H$ will depend on $k_H$. More precisely,

$$v_H = -k_H \nabla p_H = (k_H/l, 0).$$

Below we will apply the WOLS scheme, the OLS scheme and the energy norm method to determine $k_H$.

## 7.1 The WOLS scheme

Recall the definition of the WOLS method given in section 5. For the problem described above, it is straight forward to find the solution of the associated minimisation problem, see (3). More precisely, it is easy to verify that the solution is given by the harmonic average of the fine scale permeability data, i.e.

$$k_H = \frac{l \cdot h}{\sum_{i=1}^{n} \frac{|b_i|}{k_i}}.$$

Here $|b_i|$ represents the measure (area) of $b_i$.

It is well known that, in some cases, the harmonic mean tends to underestimate the flow and may overestimate the effect of low permeable zones. For instance, if $k_j = \epsilon \sim 0$ and $k_i = 1$ for $i \neq j$ then

$$k_H = \frac{\epsilon \cdot n^2}{\epsilon(n^2 - 1) + 1},$$

where we assume that $l = h = 1$. Hence $k_H$ is of order $\epsilon$, i.e. close to zero, which is not desirable in such cases.

## 7.2 The OLS scheme

Consider again the $L^2$-norms defined in (4) and (5) and the $L_2$ minimisation problem given in section 6. Clearly, if the following problem

$$\min_{k_H} \| v_h - v_H(k_H) \|_{L^2}^2, \tag{18}$$

has a unique solution, then this solution must be the upscaled permeability $k_H$ determined by the OLS scheme (if it is unique then it is not necessary to minimise the difference between the fine scale and coarse scale pressure fields).

For the simple case discussed in the introduction of this section, (18) can easily be solved by hand. The unique solution is given by

$$k_H = \frac{l}{h} \int_B \nabla p_h \cdot k_h \nabla p_h \, dx.$$

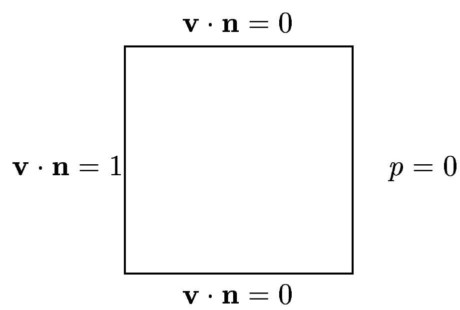


Figure 4: Suitable boundary conditions used in the "energy norm" approach to upscaling in section 7

This result coincides with the coarse scale permeability obtained by the energy dissipation principle, i.e. by requiring that

$$\int_B \nabla p_H \cdot (k_H \nabla p_H)\, dx = \int_B \nabla p_h \cdot (k_h \nabla p_h)\, dx.$$

In such simple cases the classical schemes are well known to handle the upscaling problem accurately. Thus we conclude that the OLS method provides adequate results for this kind of problems.

However, it should be mentioned that in more complex, and realistic, cases the results obtained by the OLS technique differs significantly from the upscaled permeability defined by local methods, see section 8.

### 7.3 The energy norm

Let us have a closer look at the energy norm approach to upscaling. Recall, that for the test problem discussed above, $p_H(k_H) = 1 - x/l$ and is thus independent of $k_H$. Consequently, the problem

$$\min_{k_H} \| p_h - p_H(k_H) \|_E^2,$$

is not meaningful in this case. The energy norm simply fails to define an upscaled permeability field.

It turns out that this problem can be rectified by changing the boundary conditions. Consider the boundary conditions depicted in Figure 4. In this case the solution of coarse scale pressure equation (2) is given by

$$p_H(x, y) = p_H(k_H; x, y) = -\frac{x}{k_H} + \frac{l}{k_H}$$

and is thus depending on $k_H$. Moreover, it is easy to verify that the solution of the energy norm minimisation problem is given by the arithmetic average of the fine scale permeability data

$$k_H = \frac{\int_B k_H \; dx}{|B|} = \frac{\sum_{i=1}^n k_i \cdot |b_i|}{|B|}.$$

However, estimates based on the arithmetic average tend to overestimate the flow in the reservoir. Thin barriers are not honoured sufficiently on the coarse scale.

In this section we have seen that, for a simple test problem, both the WOLS scheme and the energy norm approach to upscaling fail to produce an acceptable permeability field on the coarse mesh. Whereas the OLS technique provide adequate results.

## 8 Numerical experiments

Now we turn our attention to a series of numerical experiments illustrating the behaviour of the upscaling technique derived above. In all of the examples we have used the program packages FLUVIAL [11] and CONTSIM [13], developed at the Norwegian Computing Center, to generate the fine scale permeability data, i.e. to generate the reservoir models. These commercial software packages, used by several oil companies to model fluvial reservoirs, produce realistic input data to be applied in the flow simulators. Thus, they should provide interesting and challenging test problems.

In each case the permeability data was upscaled from a $20 \times 20 \times 20$ uniform mesh to a $10 \times 10 \times 10$ uniform grid. In addition to algorithms 1 and 2, defined in section 6.1, we applied the classical local method, introduced by Warren and Price in [16], to compute the upscaled permeability data.

Finally, we ran algorithm 1 without any bounds on the transmissibilities, i.e. $T_L^{i,j} = -\infty$ and $T_U^{i,j} = \infty$ for all $i, j$ (cf.section 6.1). In particular, accepting both negative and positive transmissibilities. The resulting coarse scale pressure equation (2) will be indefinite. More precisely, the stiffness matrix $A$, in the associated linear equation system $Ax = b$, will have both negative and positive eigenvalues. It seems like this approach is not applicable for a commercial reservoir simulator. But from a theoretical point of view, this approach is appealing. As we will see below, this method reproduces the average fine scale flow on the coarse mesh, cf. the introduction to section 6. Moreover, the production and injection rates in the wells are preserved on the coarse scale.

In all the tables below *Locale ups.* refers to the results obtained by the classical local upscaling method, and *Indef.* represents the results generated by the theoretical approach (allowing both negative and positive transmissibilities) described above. Finally, *Proj.* refers to the $L_2$ projection of the fine scale pressure (or velocity) onto the coarse grid. We would like to emphasize that the projection is the best approximation on the coarse scale of the fine scale pressure (velocity). Hence, it is not possible to obtain a smaller error than the projection error. The reader should keep this in mind while reading the following sections.

## 8.1 Example 1

We consider a fluvial reservoir containing two channels and two wells, an injector and a producer. The specified pressure in the injector and the producer is 1 and 0, respectively. Both wells have been observed in both channels. The sand gross, i.e. the volume fraction occupied by the channels, is 4%. The rest of the reservoir contains low permeable rocks.

Table 3 contains the projection errors and the errors introduced in the pressure and velocity field by the upscaling methods discussed above. Clearly,

| Alg. | $\frac{\|p_h - p_H\|_{L^2}}{\|p_h\|_{L^2}}$ | $\frac{\|v_h - v_H\|_{L^2}}{\|v_h\|_{L^2}}$ |
|---|---|---|
| Proj. | 0.0125789 | 0.708098 |
| Indef. | 0.0125924 | 0.708098 |
| Alg. 1 | 0.0389834 | 0.755025 |
| Alg. 2 | 0.0129805 | 0.709159 |
| Locale ups. | 0.0987882 | 0.872409 |

Table 3: Numerical results obtained in Example 1; Relative error in the pressure and velocity field.

the *Indef.* approach provides optimal results, in the sense that the errors are identical (neglecting the numerical round off errors) to the projection errors. The performance of algorithm 2 is comparable to that of the *Indef.* method, and is significantly better than algorithm 1. We observe that the error in the pressure field is unacceptable for the local method.

These observations are confirmed by Table 4. Only relatively small errors

| Alg. | well 1 | well 2 |
|---|---|---|
| Fine scale | 86.207 | −86.207 |
| Indef. | 86.206 | −86.207 |
| Alg. 1 | 72.914 | −72.913 |
| Alg. 2 | 84.878 | −84.877 |
| Locale ups. | 32.028 | −32.028 |

Table 4: Numerical results obtained in Example 1; Production and injection rates in the wells.

are introduced in the production and injection rates by algorithm 2. Also the results obtained by algorithm 1 are acceptable, whereas the local method fails to solve this problem.

## 8.2 Example 2

This case is almost identical to the reservoir discussed in Example 1. Recall that in Example 1 both wells where observed in both channels. In this case we move one the wells, the producer, such that it is positioned just outside the channels. More precisely, the well path of the producer constitutes of fine scale blocks which are neighbour cells to the channels.

Again, see tables 5 and 6, the *Indef.* technique provides optimal results. Both algorithms 1 and 2 are capable of solving the problem, whereas the

| Alg. | $\frac{\|p_h - p_H\|_{L^2}}{\|p_h\|_{L^2}}$ | $\frac{\|v_h - v_H\|_{L^2}}{\|v_h\|_{L^2}}$ |
|---|---|---|
| Proj. | 0.0200679 | 0.734783 |
| Indef. | 0.0200679 | 0.735052 |
| Alg. 1 | 0.0205625 | 0.888777 |
| Alg. 2 | 0.0201004 | 0.737088 |
| Locale ups. | 0.0525788 | 7.28119 |

Table 5: Numerical results obtained in Example 2; Relative error in the pressure and velocity field.

local method clearly overestimates the flow in the reservoir. It seems like

| Alg. | well 1 | well2 |
|---|---|---|
| Fine scale | 1.133 | −1.133 |
| Indef. | 1.132 | −1.133 |
| Alg. 1 | 1.088 | −1.088 |
| Alg. 2 | 1.150 | −1.151 |
| Locale ups. | 9.511 | −9.511 |

Table 6: Numerical results obtained in Example 2; Production and injection rates in the wells.

the local method "activates" the producer on the coarse scale. The flow is overestimated by a factor of $\approx 9$.

## 8.3 Example 3

As in the previous examples we consider a fluvial reservoir. This is a more complex case, the reservoir contains 18 channels, 3 wells (one injector and two producers), and the sand gross is 30%. The pressure in the injector and producers is 1 and 0.5, respectively.

According to tables 7 and 8 the general picture is very similar to the observations made in examples 1 and 2. However, in this case it is not clear that algorithm 2 provides more accurate injection and production rates in the wells than algorithm 1. But both methods produce acceptable results compared to the local upscaling technique.

| Alg. | $\frac{\|p_h - p_H\|_{L^2}}{\|p_h\|_{L^2}}$ | $\frac{\|v_h - v_H\|_{L^2}}{\|v_h\|_{L^2}}$ |
|---|---|---|
| Proj. | 0.0266725 | 0.650512 |
| Indef. | 0.0266736 | 0.660437 |
| Alg. 1 | 0.0278723 | 0.733493 |
| Alg. 2 | 0.0268452 | 0.663891 |
| Locale ups. | 0.0466163 | 23.5491 |

Table 7: Numerical results obtained in Example 3; Relative error in the pressure and velocity field.

| Alg. | well 1 | well 2 | well 3 |
|---|---|---|---|
| Fine scale | 2.300 | −1.261 | −1.040 |
| Indef. | 2.302 | −1.260 | −1.041 |
| Alg. 1 | 2.262 | −1.292 | −0.970 |
| Alg. 2 | 2.372 | −1.315 | −1.056 |
| Locale ups. | 42.9372 | −42.1198 | −0.817261 |

Table 8: Numerical results obtained in Example 3; Production and injection rates in the wells.

## 8.4 Example 4

In this experiment we want to check if the performance of the coarse scale reservoir model, generated by the OLS scheme, is stable with respect the boundary conditions. The fine scale permeability field and the positions of the wells are as in example 3. However, the pressure in the producers has been changed form 0.5 (in wells 2 and 3), to 0.3 in well 2 and 0.7 in well 3.

We did not apply algorithms 1 or 2 directly to this problem. Instead we used the transmissibilities generated by algorithm 2 in example 3 in the discretization of the coarse scale pressure equation (2). Only the well data was changed.

Tables 9 and 10 confirms that the OLS scheme is stable with respect to this kind of changes in the boundary conditions. Again the locale upscaling

| | $\frac{\|p_h - p_H\|_{L^2}}{\|p_h\|_{L^2}}$ | $\frac{\|v_h - v_H\|_{L^2}}{\|v_h\|_{L^2}}$ |
|---|---|---|
| Proj. | 0.0311073 | 0.637436 |
| OLS | 0.0311965 | 0.656585 |
| Locale ups. | 0.0611272 | 30.7218 |

Table 9: Numerical results obtained in Example 4; Relative error in the pressure and velocity field.

method fails to handle the problem. The effect of decreasing and increasing the pressure in wells 2 and 3 can easily be observed by compare the fine scale data (or OLS data) in tables 8 and 10 (The production in well 2 has increased and the production in well 3 has decreased).

| Scale | well 1 | well 2 | well 3 |
|---|---|---|---|
| Fine scale | 2.388 | −1.766 | −0.622 |
| OLS | 2.476 | −1.843 | −0.632 |
| Locale ups. | 59.4518 | −59.0378 | −0.413872 |

Table 10: Numerical results obtained in Example 4; Production and injection rates in the wells.

# 9 Conclusions

We have developed a new upscaling technique for computing the permeability on a coarse scale. The method is applicable whenever traditional locale methods fail to compute an acceptable permeability field, e.g. upscaling of blocks close to wells and cases involving heterogeneities on the coarse grid block scale. In such cases, is it impossible to compute an effective permeability based on locale observations of the pressure and velocity fields. The properties of the coarse reservoir model will simply depend heavily on the global flow pattern.

The basic idea behind the new method is to try to minimise the errors, introduced by the upscaling process, in the pressure and velocity functions. We have discussed three different norms for measuring these errors; the energy norm, the inverse energy norm (WOLS) and the $L_2$ norm (OLS method).

The energy norm and WOLS scheme have nice mathematical properties. However, in some cases they fail to preserve important flow properties on the coarse scale.

The $L_2$ norm approach to upscaling seems to be promising. More precisely, in most coarse grid blocks the total mass flux over each coarse grid block interface is preserved on the coarse mesh. Leading to very accurate production and injection rates in the wells. Moreover, the associated minimsation problem can be solved very efficiently. Traditional, and computational expensive, optimization algorithms are not needed. However, the solution of the fine scale pressure equation is required. It turns out that, in view of modern numerical methods for elliptic differential equations, the efficiency of the OLS scheme is comparable to the performance of the traditional locale methods.

Furthermore, the paper provides a mathematical analysis of the new OLS scheme, and possible extensions to handle non-diagonal permeability tensors and multiphase (relperm) cases. These features should make the method applicable to a wide range of practical problems.

Finally, through a series of analytical examples and numerical experiments we have seen that the OLS scheme produce accurate results for several test problems. Including cases where locale methods fail to produce an acceptable upscaled permeability field.

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