A penalty scheme for solving American option problems

Bjørn Fredrik Nielsen, Ola Skavhaug,
Aslak Tveito

University of Oslo
and
The Norwegian Computing Center

Outline

- Options?
- American put, model problem.
- Penalty method.
- Numerical experiments.
- Multi-asset options.
- Conclusion.

Options?

- Option, a contract that gives the buyer the right (but no obligation) to buy (sell) an asset for a prescribed price at a prescribed expire date.
- European, American, Asian, Exotic, Barrier and Multi-asset options.
- European, exercise only permitted at expire.
- American, exercise permitted at any time during the life of the option.

Options?, continued ...

- No arbitrage, risk-free interest rate, continuous trading, etc.
- Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0.$$

  - $P = P(S,t)$; risk-neutral price of the option.
  - S; underlying asset.
  - r; interest rate.
  - σ; volatility.
- European, fixed solution domain.
- American, moving boundary.

American put

- For $S > \bar{S}(t)$ and $0 \leq t < T$

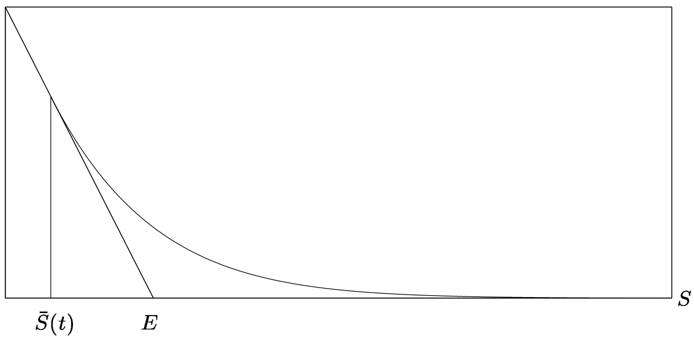
$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0.$$

- For $0 \leq S < \bar{S}(t)$

$$P(S,t) = E - S.$$

- $\bar{S}(t)$; unknown moving boundary.

American put, continued ...



- E; Exercise price.
- No arbitrage, constraint

$$\underline{P(S,t) \geq \max(E-S,0)}.$$

- P, $\frac{\partial P}{\partial S}$ continuous.
- Final conditions (backwards in time!)

$$P(S,T) = \max(E-S,0),$$
$$\bar{S}(T) = E.$$

Penalty method

- Recall the constraint

$$P(S,t) \geq \max(E - S, 0).$$

- Zvan, Forsyth and Vetzal (1998);
  - Discrete P gets close to the constraint.
  - Add a "LARGE" number to the discrete equations.
  - "Push" the appr. solution away from the constraint.
- Our approach; Add a continuous penalty term to the Black-Scholes equation.

Penalty method, continued ...

- For $S \geq 0$ and $t \in [0, T)$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP + \frac{\epsilon C}{P + \epsilon - (E - S)} = 0.$$

  - $C \geq rE$ positive constant.
  - $0 < \epsilon \ll 1$.
  - Nonlinear PDE posed on a fixed domain.

Penalty method, continued ...

The penalty term

$$\frac{\epsilon C}{P + \epsilon - (E - S)}$$

is

- of order ϵ if $P \gg (E - S)$.
- $\approx C \geq rE$ as $P \to (E - S)$.

Penalty method, continued ...

- Explicit scheme (backwards in time!);

$$\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \frac{P_{j+1}^n - 2P_j^n + P_{j-1}^n}{(\Delta S)^2}$$
$$+ rS_j \frac{P_{j+1}^n - P_j^n}{\Delta S} - rP_j^n + \frac{\epsilon C}{P_j^n + \epsilon - q_j} = 0.$$

- **Theorem 1**

For all $C \geq rE$,

$$P_j^n \geq \max(E - S_j, 0),$$

provided that

$$\Delta t \leq \frac{(\Delta S)^2}{\sigma^2 S_\infty^2 + rS_\infty(\Delta S) + r(\Delta S)^2 + \frac{C}{\epsilon}(\Delta S)^2}.$$

Penalty method, continued ...

- Fully-implicit (nonlinear equations);

$$\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \frac{P_{j-1}^{n-1} - 2P_j^{n-1} + P_{j+1}^{n-1}}{(\Delta S)^2}$$

$$+ rS_j \frac{P_{j+1}^{n-1} - P_j^{n-1}}{\Delta S} - rP_j^{n-1} + \frac{\epsilon C}{P_j^{n-1} + \epsilon - q_j} = 0.$$

- **Theorem 2**

  For all $C \geq rE$,

$$P_j^n \geq \max(E - S_j, 0).$$

- No condition on Δt required!

Penalty method, continued ...

- Semi-implicit (linear equations);

$$\frac{P_j^n - P_j^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \frac{P_{j-1}^{n-1} - 2P_j^{n-1} + P_{j+1}^{n-1}}{(\Delta S)^2}$$
$$+ rS_j \frac{P_{j+1}^{n-1} - P_j^{n-1}}{\Delta S} - rP_j^{n-1} = -\frac{\epsilon C}{P_j^n + \epsilon - q_j}.$$

- **Theorem 3**

  For all $C \geq rE$,

$$P_j^n \geq \max(E - S_j, 0),$$

  provided that

$$\Delta t \leq \frac{\epsilon}{rE}.$$

Numerical experiments

- Model parameters;

$$
\begin{aligned}
r &= 0.1, \\
\sigma &= 0.2, \\
E &= 1, \\
T &= 1.
\end{aligned}
$$

- Reference solution; Implicit Front-Fixing.

Numerical experiments, continued ...

- Explicit;

ϵ	L_1	L_2	L_∞	H_1	CPU-time
10^{-1}	$2.50 \cdot 10^{-2}$	$2.61 \cdot 10^{-2}$	$4.23 \cdot 10^{-2}$	$1.00 \cdot 10^{-1}$	$129.5s$
10^{-2}	$5.04 \cdot 10^{-3}$	$6.31 \cdot 10^{-3}$	$1.32 \cdot 10^{-2}$	$4.05 \cdot 10^{-2}$	$129.5s$
10^{-3}	$6.22 \cdot 10^{-4}$	$9.49 \cdot 10^{-4}$	$2.50 \cdot 10^{-3}$	$1.10 \cdot 10^{-2}$	$129.6s$
10^{-4}	$1.18 \cdot 10^{-4}$	$1.51 \cdot 10^{-4}$	$3.02 \cdot 10^{-4}$	$2.50 \cdot 10^{-3}$	$130.2s$

- Fully-implicit;

ϵ	L_1	L_2	L_∞	H_1	CPU-time
10^{-1}	$2.50 \cdot 10^{-2}$	$2.61 \cdot 10^{-2}$	$4.23 \cdot 10^{-2}$	$1.00 \cdot 10^{-1}$	$7.8s$
10^{-2}	$5.03 \cdot 10^{-3}$	$6.30 \cdot 10^{-3}$	$1.32 \cdot 10^{-2}$	$4.05 \cdot 10^{-2}$	$7.8s$
10^{-3}	$6.19 \cdot 10^{-4}$	$9.45 \cdot 10^{-4}$	$2.49 \cdot 10^{-3}$	$1.10 \cdot 10^{-2}$	$7.8s$
10^{-4}	$1.20 \cdot 10^{-4}$	$1.54 \cdot 10^{-4}$	$2.99 \cdot 10^{-4}$	$2.50 \cdot 10^{-3}$	$8.4s$

- Semi-implicit;

ϵ	L_1	L_2	L_∞	H_1	CPU-time
10^{-1}	$2.50 \cdot 10^{-2}$	$2.61 \cdot 10^{-2}$	$4.23 \cdot 10^{-2}$	$1.00 \cdot 10^{-1}$	$2.8s$
10^{-2}	$5.03 \cdot 10^{-3}$	$6.31 \cdot 10^{-3}$	$1.32 \cdot 10^{-2}$	$4.05 \cdot 10^{-2}$	$2.8s$
10^{-3}	$6.21 \cdot 10^{-4}$	$9.48 \cdot 10^{-4}$	$2.49 \cdot 10^{-3}$	$1.10 \cdot 10^{-2}$	$2.8s$
10^{-4}	$1.19 \cdot 10^{-4}$	$1.52 \cdot 10^{-4}$	$3.01 \cdot 10^{-4}$	$2.49 \cdot 10^{-3}$	$2.8s$

Multi-asset options

- Assets; S_1, S_2.
- Option price; $P = P(S_1, S_2, t)$.
- Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_1^2 \, S_1^2 \, \frac{\partial^2 P}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 \, S_2^2 \, \frac{\partial^2 P}{\partial S_2^2}$$
$$+\rho\sigma_1\sigma_2 \, S_1 S_2 \, \frac{\partial^2 P}{\partial S_2 \partial S_1}$$
$$+rS_1 \, \frac{\partial P}{\partial S_1} + rS_2 \, \frac{\partial P}{\partial S_2} - rP = 0.$$

- ρ; correlation between the assets.

Multi-asset options, continued ...

- Payoff function at expire

$$\phi(S_1, S_2) = \max\left(E - (\alpha_1 S_1 + \alpha_1 S_1), 0\right).$$

- American options → constraint

$$P(S_1, S_2, t) \geq \phi(S_1, S_2).$$

- Penalty term

$$\frac{\epsilon C}{P + \epsilon - (E - (\alpha_1 S_1 + \alpha_1 S_1))}.$$

- Analysis,

$$C \geq rE.$$

Multi-asset options, continued ...

- We define explicit, fully-implicit and semi-implicit schemes.
- $\rho = 0$, i.e. independent assets, we prove that the constraint is fulfilled.
- $\rho \neq 0$, numerical experiments indicate that the constraint is satisfied.
- Fine meshes, the semi-implicit scheme is preferable.

Conclusion

- Both American single- and multi- asset options can be priced efficiently by penalty methods.
- Explicit scheme; easy to implement, inefficient.
- Fully-implicit scheme; "hard to implement", efficient.
- Semi-implicit; "easy to implement", efficient.