The proposed algorithms for eliminating cuts in the provability calculus GLS do not terminate

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Provability Logic

The logical system characterizing provability logic correspond to a modal logic K4, the logic consisting of:

- $\mathrm{Bew}[A \to B] \to (\mathrm{Bew}[A] \to \mathrm{Bew}[B])$ (K),
- $\mathrm{Bew}[A] \to \mathrm{Bew}[\mathrm{Bew}[A]]$ (4),
- if $\pi \vdash A$ then $\pi^* \vdash \mathrm{Bew}[A]$ (neccesitation)
- fixpoint-property on $\mathrm{Bew}[]$.
- complete set of axioms of propositional logic
- if $\pi_1 \vdash A$ and $\pi_2 \vdash A \to B$ then $\pi^* \vdash B$ (modus ponens)

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The language

Definition 1 *The language L_{GLS} is defined as;*

1. $\perp$, $P_i \in ATO$ for all $i \in N$
2. If A, B $\in L_{GLS}$ then $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $\Box A \in L_{GLS}$

   From now on we interpret $\Box$ as $\mathrm{Bew}[]$, and we limit ourselves to propositional logic.

Previous work

1. Leivant, Daniel: “On the Proof Theory of the Modal Logic for Arithmetic”, Journal of Symbolic Logic, 1981, p. 531 - 538
2. Sambin, Giovanni and Silvio Valentini: “The modal logic of provability: The sequential approach” Journal of Philosophical Logic, 1982, p. 311 -342
3. Valentini, Silvio: “The modal logic of provability: Cut-elimination” Journal of Philosophical Logic, 1983, p.471 - 476

The agenda

- Provability Logic formulated as Gentzen type sequential calculus
- Modus Ponens - an instance of the more general cut rule
- Cut introduces “magic” to the proof
- Important to understand cut-elimination

Knowledge about proofs

Definition 2 *Let* π *be a proof. Then* $|\Re(\pi)|$, *where* $\Re$ *is a n-ary inference rule, defined by recursion on* π *as follows:*

1. If π is the proof of an axiom then $|\Re(\pi)| = 0$
2. If the last rule applied in π is $\Re$, and the immediate subproofs of π are $\pi_1, \ldots, \pi_n$, then $|\Re(\pi)| = 1 + \sum_{i=1}^{n} |\Re(\pi_i)|$.
3. If the last rule applied in π is different from $\Re$, and the immediate subproofs of π are $\pi_1, \ldots, \pi_n$, then $|\Re(\pi)| = \sum_{i=1}^{n} |\Re(\pi_i)|$.

Definition 3 *A connected region μ is defined as:*

1. If μ is a sequent then it is a connected region.
2. If $\mu_1 \ldots \mu_n$ are connected regions and $\Re$ is an inference rule such that $\Gamma \vdash \Delta$ is a logical consequence of $\mu_1 \ldots \mu_n$ then

$$\frac{\mu_1 \quad \ldots \quad \mu_n}{\Gamma \vdash \Delta}\,\Re$$

   is a connected region.

Definition 4 *If μ is a connected region, then inf(μ) is the sequent in the root of the region μ, sup(μ) is the set of sequences not following logically from any sequents in the region.*

Definition 5 *Let μ_1 and μ_2 be two connected regions such that there exists a y $\in sup(\mu_2)$ such that y is a logical consequence of $inf(\mu_1)$*

Then we write the region-concatination of μ_1 followed by μ_2 as;

$$\begin{array}{c}\mu_2 \\ | \\ \mu_1\end{array}$$

that itself is a new region.

Definition 6 *Let μ be as in the previous definition, then:*

$$(\mu)^1 = \mu \qquad (\mu)^{n+1} = \begin{array}{c}\mu \\ | \\ (\mu)^n\end{array}$$

Transformation on regions

Definition 7 *Let μ_1 and μ_2 be connected regions. μ_1 is* **transferable** *to μ_2, $\mu_1 \longmapsto \mu_2$ iff $inf(\mu_1) = inf(\mu_2)$.*

Definition 8 *Let μ_1 and μ_2 be connected regions. μ_1 is* **reducable** *to μ_2 (we write $\mu_1 \hookrightarrow \mu_2$) iff $\mu_1 \longmapsto \mu_1$ and $d(\mu_2) \leq d(\mu_1)$. μ_1 is* **strictly reducable** *to μ_2 if $\mu_1 \hookrightarrow \mu_2$ and $d(\mu_2) < d(\mu_1)$.*

Lemma 1 $\mu\mu \longmapsto \mu$

Lemma 2 $(\mu)^n \longmapsto \mu$

If μ and κ both are regions, we write $\kappa(\mu)$ when μ is a subregion in κ.

Lemma 3 *If $\mu_1 \longmapsto \mu_2$ then $\kappa(\mu_1) \longmapsto \kappa(\mu_2)$.*

Proof: Obvious.

Regions and modality

Some regions in GLS proofs are significant and deserve specific names:

Definition 9 *A connected region* μ *is* **modal free region**, *MFR, iff* $|GLR(\mu)| = 0$.

Definition 10 *The largest modalfree region relative to a proof* π *is the* **modal free closure** *of* π, $MFC(\pi)$, *with respect to the conclusion of* π.

$MFC(\pi)$ is a connected region.

Definition 11 *An upper bound of a MFC is the set of supremums of the modal free closure of the MFC, in other words* $ub(MFC(\pi)) = sup(MFC(\pi))$.

Definition 12 *If the upper bound of* $MFC(\pi) = \{S_1, \ldots S_n\}$, *the* **exterior** *of the* $MFC(\pi)$ *are the proofs* $\pi_1, \ldots, \pi_n$ *for respectively* $S_1, \ldots, S_n$.

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Löb's theorem proved automatically

Theorem prover VALSAM's proof of Löb's theorem:

The sequent $\vdash (\Box((\Box A) \to A)) \to (\Box A)$ is a theorem in GLS'.

The tree looks like this:

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{\Box((\Box A) \to A), \Box A, A \vdash A} \qquad \cfrac{\cfrac{\cfrac{A \vdash A}{(\Box A) \to A, A, \Box((\Box A) \to A), \Box A, \Box A \vdash A}}{\Box((\Box A) \to A), \Box A \vdash \Box A}}{\Box((\Box A) \to A), \Box A \vdash A, \Box A}}{(\Box A) \to A, \Box((\Box A) \to A), \Box A \vdash A}}{\Box((\Box A) \to A) \vdash \Box A}}{\vdash (\Box((\Box A) \to A)) \to (\Box A)}$$

Eliminating cuts in GLS

Problems:

- Modal rules might cause loops in proofs
- Every application of GLR gives rise to a new possible world
- GLR gives new modal formulas
- There is a cutfree proof of every theorem in GLS since GLS' is a cut free calculus proven to be complete in the previous chapter. Hence there should be an algorithm for systematically eliminate every cut given a proof π in GLS.

Leivant's proposal

Leivant proposed a solution to GLS where he considered proofs like

$$\dfrac{\dfrac{\begin{array}{c}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{array}}{\Box\Gamma \vdash \Box\varphi}\,GLR \qquad \dfrac{\begin{array}{c}\pi_R\\ |\\ \Delta, \Box\Delta, \varphi, \Box\varphi, \Box A \vdash A\end{array}}{\Box\Delta, \Box\varphi \vdash \Box A}\,GLR}{\Box\Gamma, \Box\Delta \vdash \Box A}\,cut \qquad fig.1$$

Definition 13 *We say that a proof π is in Sambin Normal Form (SNF) if and only if the last inference rule in π is the cut-rule, and the last inference-rule applied in each subtree is GLR.*

Now the cut over that is in SNF and by $MOVE_{Lei}$ it is reduced to:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{array}}{\Box\Gamma \vdash \Box\varphi}\,GLR \qquad \dfrac{\dfrac{\begin{array}{c}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{array} \quad \begin{array}{c}\pi_R\\ |\\ \Delta, \Box\Delta, \varphi, \Box\varphi, \Box A \vdash A\end{array}}{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box\varphi, \Box\varphi, \Box A \vdash A}\,cut_2}{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box\varphi, \Box A \vdash A}\,contr}{\Gamma, \Box\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}\,cut_1}{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}\,contr}{\Box\Gamma, \Box\Delta \vdash \Box A}\,GLR$$

Valentini's argument

(1) If $|GLR(\pi_L| = 0$ then $\Box\varphi$ is introduced by weakening and we can eliminate it from the tree. Then we apply the move VAL-I:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_L^*\\ |\\ \Gamma, \Box\Gamma \vdash \varphi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{matrix}}{\Box\Gamma \vdash \Box\varphi}\,GLR \qquad \begin{matrix}\pi_R\\ |\\ \Delta, \Box\Delta, \varphi, \Box\varphi, \Box A \vdash A\end{matrix}}{\Box\Gamma, \Delta, \Box\Delta, \varphi, \Box A \vdash A}\,cut_2}{\Gamma, \Box\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}\,cut_1}{\dfrac{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}{\Box\Gamma, \Box\Delta \vdash \Box A}\,GLR}\,contr$$

Where π_L^* is like π_L except that every introduction of $\Box\varphi$ by weakening is removed. (2) If $|GLR(\pi_L| > 0$ then the core of Valentini's reduction is VAL-II (core):

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{matrix}}{\Box\Gamma \vdash \Box\varphi}\,GLR \qquad \begin{matrix}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{matrix}}{\Gamma, \Box\Gamma, \Box\Gamma \vdash \varphi}\,cut_{Val} \qquad \dfrac{\dfrac{\begin{matrix}\pi_L\\ |\\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi\end{matrix}}{\Box\Gamma \vdash \Box\varphi} \qquad \begin{matrix}\pi_R\\ |\\ \Delta, \Box\Delta, \varphi, \Box\varphi, \Box A \vdash A\end{matrix}}{\Box\Gamma, \Delta, \Box\Delta, \varphi, \Box A \vdash A}\,cut_2}{\Gamma, \Box\Gamma, \Box\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}\,cut_1}{\dfrac{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}{\Box\Gamma, \Box\Delta \vdash \Box A}\,GLR}\,contr$$

Valentini's response to $|GLR(\pi_L)| > 0$

$$
\dfrac{
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\dfrac{
\begin{array}{c}
\dfrac{\Box C \vdash \Box C}{\Box\varphi, \Box C, \Box\Gamma^* \vdash \Box C} \\ | \\ \varrho_j^* \\ | \\ \Box C, \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi
\end{array}
}{
\dfrac{C, \Box C, \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi}{\Box C, \Box\Gamma \vdash \Box\varphi}
}
\quad
\begin{array}{c}
\pi_j \\ | \\ \dfrac{\varphi, \Box\varphi, \Gamma^*, \Box\Gamma^* \Box C \vdash C}{\Box\varphi, \Box\Gamma^* \vdash \Box C} \\ | \\ \varrho_j \\ | \\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi
\end{array}
}{\Box C, \Gamma, \Box\Gamma, \Box\Gamma \vdash \varphi}
\quad
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{\Box C \vdash \Box C}{\Box\varphi, \Box C, \Box\Gamma^* \vdash \Box C} \\ | \\ \varrho_j^* \\ | \\ \Box C, \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi
\end{array}
}{
\dfrac{C, \Box C, \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi}{\Box C, \Box\Gamma \vdash \Box\varphi}
}
\quad
\begin{array}{c}
\pi_j \\ | \\ \varphi, \Box\varphi, \Gamma^*, \Box\Gamma^*, \Box C \vdash C
\end{array}
}{\varphi, \Box\Gamma, \Gamma^*, \Box\Gamma^*, \Box C, \Box C \vdash C}
}{
\dfrac{\Gamma, \Box\Gamma, \Box\Gamma, \Box\Gamma, \Gamma^*, \Box\Gamma^*, \Box C, \Box C, \Box C \vdash C}{
\dfrac{\Gamma, \Box\Gamma, \Gamma^*, \Box\Gamma^*, \Box C \vdash C}{
\dfrac{\Box\Gamma, \Box\Gamma^*, \vdash \Box C}{\Box\varphi, \Box\Gamma, \Box\Gamma^*, \vdash \Box C}
}
}
}
$$

(CCC)

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and then we write the general move VAL-II as:

$$
\dfrac{
\dfrac{
 \dfrac{\dfrac{\begin{array}{c} CCC \\ | \\ \varrho_j^{**} \\ | \\ \Gamma, \Box\Gamma, \Box\Gamma, \Box\varphi \vdash \varphi \end{array}}{\Gamma, \Box\Gamma, \Box\varphi \vdash \varphi}}{\Box\Gamma \vdash \Box\varphi}
 \quad
 \begin{array}{c} \pi_j \\ | \\ \dfrac{\varphi, \Box\varphi, \Gamma^*, \Box\Gamma^*, \Box C \vdash C}{\Box\varphi, \Box\Gamma^* \vdash \Box C} \\ | \\ \varrho_j \\ | \\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi \end{array}
}{\Gamma, \Box\Gamma, \Box\Gamma \vdash \varphi}\, cut_{Val}
\quad
\dfrac{
 \dfrac{\begin{array}{c} \pi_j \\ | \\ \dfrac{\varphi, \Box\varphi, \Gamma^*, \Box\Gamma^*, \Box C \vdash C}{\Box\varphi, \Box\Gamma^* \vdash \Box C} \\ | \\ \varrho_j \\ | \\ \Gamma, \Box\Gamma, \Box\varphi \vdash \varphi \end{array}}{\Box\Gamma \vdash \Box\varphi}
 \quad
 \begin{array}{c} \pi_r \\ | \\ \Delta, \Box\Delta, \varphi, \Box\varphi, \Box A \vdash A \end{array}
}{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \varphi, \Box A \vdash A}\, cut_2
}{
\dfrac{\dfrac{\Gamma, \Box\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}{\Gamma, \Box\Gamma, \Delta, \Box\Delta, \Box A \vdash A}\, contr}{\Box\Gamma, \Box\Delta \vdash \Box A}
}\, cut_1
$$

where ϱ_j^* is like ϱ_j except $\Box C$ is added and ϱ_j^{**} is like ϱ_j except $\Box\Gamma$ is added.

Definition 14 *We call the j'th GLR application that plays the significant role the VAL-II reduction for the* **left principal GLR**, *(LP_{GLR}).*

SHIFT arguments

$$
\dfrac{\dfrac{\dfrac{\pi_L}{\Gamma_{11} \vdash \Delta_{11}}}{\Gamma_{11} \vdash \Delta_{11}, A}\,weak \quad \vdots\ \varrho_1 \quad \Gamma_1 \vdash \Delta_1, A \qquad \begin{matrix}\pi_R\\ \vdots \\ \Gamma_2, A \vdash \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,cut \quad \longmapsto \quad \dfrac{\begin{matrix}\pi_L\\ \vdots\\ \Gamma_{11} \vdash \Delta_{11}\\ \vdots\\ \varrho_1 \ominus succed(A)\\ \vdots\\ \Gamma_1 \vdash \Delta_1\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}
$$

We say in this case that we apply a **shift argument** on the left branch.For the right-branch we have similar argument.

Addition and subtraction on trees

Definition 15

If $\Gamma \vdash \Delta$ *is a sequent S, then ant(S)* $= \Gamma$, $succ(S) = \Delta$.

$ant(S) \oplus A = \Gamma \cup \{A\}$

$succ(S) \oplus A = \Delta \cup \{A\}$

if $A \in \Gamma$ *then* $ant(S) \ominus A = \Gamma \setminus \{A\}$

$A \in \Delta$ *then* $succ(S) \ominus A = \Delta \setminus \{A\}$

We write $ant_R((\varrho)$ to denote the set of antecedents in ϱ and $succ_R(\varrho)$ to denote the set of succedents in ϱ. Algebra on regions is defined as:

Addition and subtraction on trees

Definition 16 *Let ϱ be a connected region, let A be a formula then $\varrho \oplus\ antec(A)$ is defined as:*

1. If $d(\varrho) = 0$ then ϱ is a sequent and $\varrho\ \oplus\ antec(A) = ant(\varrho) \cup A \vdash succ(\varrho)$.

2. If $d(\varrho)\ >\ 0$ and the last rule is a n-branching rule with immediate subregions $\varrho_1 \ldots \varrho_n$, then

$$\mu \oplus antec(A) = \qquad \frac{\varrho_1 \oplus antec(A) \qquad \ldots \qquad \varrho_n \oplus antec(A)}{ant(inf(\varrho)) \cup A \vdash succ(inf(\varrho))}$$

Then we define $\varrho \oplus\ succed(A)$, $\varrho \ominus\ antec(A)$ and $\varrho \ominus\ succed(A)$ likewise. This gives a language for expressing precicely what ϱ_j^* and ϱ_j^{**} are, $\varrho_j^* = \varrho_j \oplus antec(\Box C)$ and $\varrho_j^{**} = \varrho_j \oplus antec(\Box\Gamma)$.

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Region multiplication

Definition 17 *Let S_1 and S_2 be sequents, then* $S_1 \oplus S_2 = ant(S_1) \cup ant(S_2) \vdash succ(S_1) \cup succ(S_2)$

Definition 18 *Let ϱ_1 and ϱ_2 be connected regions,* $|cut(\varrho_1)| = |cut(\varrho_2)| = 0$ *and* $|GLR(\varrho_1)| = |GLR(\varrho_2)| = 0$*. By recursion on both $d(\varrho_1)$ and $d(\varrho_2)$ we define $\bigotimes$.*

1. If $d(\varrho_1) = 0$ and $d(\varrho_2) = 0$ then $\varrho_1 \bigotimes \varrho_2 = inf(\varrho_1) \oplus inf(\varrho_2)$.

2. Let $d(\varrho_1) = 0$ and $d(\varrho_2) > 0$ and the last rule applied in ϱ_2 is a n branching rule $\Re$ with immediate subregion $\varrho_2^1 \ldots \varrho_2^n$ then

$$\varrho_1 \bigotimes \varrho_2 = \frac{\varrho_1 \bigotimes \varrho_2^1 \quad \ldots \quad \varrho_1 \bigotimes \varrho_2^n}{inf(\varrho_1) \oplus inf(\varrho_2)} \Re$$

3. If $d(\varrho_1), d(\varrho_2) > 0$ and the last rule applied in ϱ_2 is a branching rule $\Re$ with immediate subregions $\varrho_1^1 \ldots \varrho_1^n$ then

$$\varrho_1 \bigotimes \varrho_2 = \frac{\varrho_1^1 \bigotimes \varrho_2 \quad \ldots \quad \varrho_1^n \bigotimes \varrho_2}{inf(\varrho_1) \oplus inf(\varrho_2)} \Re$$

Properties of region multiplication

Lemma 4 *If* ϱ_1 *and* ϱ_2 *are connected regions in* GLS^-*, then* $\varrho_1 \bigotimes \varrho_2 \longmapsto \varrho_2 \bigotimes \varrho_1$

Lemma 5 *Let* ϱ_L *and* ϱ_R *are connected regions in* GLS^- *with n binary branching rules in* ϱ_L *and m binary branching rules in* ϱ_R*. Let the trunc regions below the leaf nodes are* ϱ_L^i *and* ϱ_R^j *for* $i \leq n$ *and* $i \leq m$*. Let the leaf nodes contain the sequents* S_L^i *and* S_L^j*. Then we have*

1. $\varrho_L \bigotimes \varrho_R$ contains $(n+1) \times (m+1)$ leaf nodes.
2. The upmost trunc regions in $\varrho_L \bigotimes \varrho_R$ are $\varrho_L^i \bigotimes \varrho_R^j$.
3. The leaf nodes in $\varrho_L \bigotimes \varrho_R$ contains the sequents $S_L^i \oplus S_L^j$.

The generality of the ROOT-move

Definition 19 *Let* $CUT(\pi_1, \pi_2)$ *denote the region:*

$$\dfrac{\begin{array}{c}\pi_1\\ |\\ \Gamma_1, \Box\varphi \vdash \Delta_1\end{array} \qquad \begin{array}{c}\pi_2\\ |\\ \Gamma_2 \vdash \Box\varphi, \Delta_2\end{array}}{}$$

Theorem 1 *(Sambin Normal Form Theorem) If* π *is a proof in GLS containing only one cut as the final inference rule and* π_L *and* π_R *have* MFC°*'s* ϱ_L *and* ϱ_R *with respectively n and m leaf nodes, then there is a proof* π_{ROOT} *such that*

$$\dfrac{\begin{array}{c}\sum_{i=1}^{n} \sum_{j=1}^{m} CUT(\pi_L^i, \pi_R^j)\\ (\varrho_L \ominus succed(\Box\varphi)) \bigotimes (\varrho_R \ominus antec(\Box\varphi))\\ |\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

where $CUT(\pi_L^i, \pi_R^j)$ *are on SNF for all* $i \leq n$ *and* $j \leq m$*.*

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Reasoning about concrete proof-objects

The proof ε:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_1 \vdash P_1}{P_1 \wedge P_2 \vdash P_1}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box P_1 \vdash P_1}}{\Box(P_1 \wedge P_2) \vdash \Box P_1}}{P_2, \Box(P_1 \wedge P_2) \vdash \Box P_1} \quad \dfrac{P_2 \vdash P_2}{\Box(P_1 \wedge P_2), P_2, \neg P_2 \vdash}}{P_2, \Box P_1 \to \neg P_2, \Box(P_1 \wedge P_2) \vdash}}{P_2, \Box P_2, \Box P_1 \to \neg P_2, \Box(\Box P_1 \to \neg P_2), \Box(P_1 \wedge P_2) \vdash P_1 \wedge P_2}}{\Box P_2, \Box(\Box P_1 \to \neg P_2) \vdash \Box(P_1 \wedge P_2)} \quad \dfrac{\dfrac{\dfrac{\dfrac{P_2 \vdash P_1}{P_1 \vdash P_1 \vee P_3}}{P_1 \wedge P_2 \vdash P_1 \vee P_3}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Box(P_1 \wedge P_2) \vdash \Box(P_1 \vee P_3)}}{\Box P_2, \Box(\Box P_1 \to \neg P_2) \vdash \Box(P_1 \vee P_3)}$$

Let Σ be the subproof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_1 \vdash P_1}{P_1 \wedge P_2 \vdash P_1}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box P_1 \vdash P_1}}{\Box(P_1 \wedge P_2) \vdash \Box P_1}}{P_2, \Box(P_1 \wedge P_2) \vdash \Box P_1} \quad \dfrac{\dfrac{P_2 \vdash P_2}{P_2, \neg P_2 \vdash}}{\Box(P_1 \wedge P_2), P_2, \neg P_2 \vdash}}{P_2, \Box P_1 \to \neg P_2, \Box(P_1 \wedge P_2) \vdash}}{P_2, \Box P_2, \Box P_1 \to \neg P_2, \Box(\Box P_1 \to \neg P_2), \Box(P_1 \wedge P_2) \vdash P_1 \wedge P_2}$$

Leivant's reduction sequences of ε

$$\varepsilon \longmapsto \pi^{\varepsilon}_{Lei_1} \longmapsto \pi^{\varepsilon}_{Lei_2} \longmapsto \pi^{\varepsilon}_{Lei_3} \longmapsto \pi^{\varepsilon}_{Lei_4} \longmapsto \pi^{\varepsilon}_{Lei_5} \longmapsto \pi^{\varepsilon}_{Lei_6} \longmapsto \pi^{\varepsilon}_{Lei_7}$$

$$= \begin{array}{c} \pi^{\varepsilon_{root}}_{Lei} \\ | \\ \varrho_{Lei} \\ | \\ \mu_{Lei} \end{array} \longmapsto \begin{array}{c} \pi^{\varepsilon_{red}}_{Lei} \\ | \\ \varrho_{Lei} \\ | \\ \mu_{Lei} \end{array} \longmapsto \begin{array}{c} \pi^{\varepsilon_{shift}}_{Lei} \\ | \\ \varrho_{Lei} \\ | \\ \mu_{Lei} \end{array} \longmapsto \begin{array}{c} \pi^{\varepsilon_{root}}_{Lei} \\ | \\ \varrho_{Lei} \\ | \\ \varrho_{Lei} \\ | \\ \mu_{Lei} \end{array}$$

- $|Cut(\pi^{\varepsilon_{root}}_{Lei}\,\varrho_{Lei})| = 1$
- $|Cut(\pi^{\varepsilon_{red}}_{Lei}\,\varrho_{Lei})| = 2$
- $|Cut(\pi^{\varepsilon_{shift}}_{Lei}\,\varrho_{Lei})| = 1$

The proof $\pi^{\varepsilon}_{Lei_7}$ or $\pi^{\varepsilon_{root}}_{Lei}\varrho_{Lei}\mu_{Lei}$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \begin{matrix}\Sigma \\ | \\ \Box\Pi_1 \vdash \Box(P_1 \wedge P_2)\end{matrix}
            \qquad
            \dfrac{\dfrac{\dfrac{P_1 \vdash P_1}{P_1 \wedge P_2 \vdash P_1}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box P_1 \vdash P_1}}{\Box(P_1 \wedge P_2) \vdash \Box P_1}
          }{
            \dfrac{\Box\Pi_1 \vdash \Box P_1}{\Box\Pi_1, P_2, \vdash \Box P_1}
          }
          \qquad
          \dfrac{\dfrac{P_2 \vdash P_2}{P_2, \neg P_2 \vdash}}{\Box\Pi_1, P_2, \neg P_2 \vdash}
        }{
          \Box\Pi_1, P_2, \Box P_1 \to \neg P_2, \vdash
        }
      }{
        \Box\Pi_1, P_2, \Box P_2, \Box P_1 \to \neg P_2, \Box(\Box P_1 \to \neg P_2), \Box P_1 \vdash P_1
      }
    }{
      \dfrac{\dfrac{\Pi_1, \Box\Pi_1, \Box P_1 \vdash P_1}{\Box\Pi_1 \vdash \Box P_1}}{P_2, \Box\Pi_1 \vdash \Box P_1}
    }
    \qquad
    \dfrac{\dfrac{P_2 \vdash P_2}{P_2, \neg P_2 \vdash}}{P_2, \neg P_2, \Box\Pi_1 \vdash}
  }{
    P_2, \Box P_1 \to \neg P_2, \Box\Pi_1 \vdash
  }
}{
  \dfrac{\dfrac{\Pi_1, \Box\Pi_1, \Box\Pi_1, \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}{\Pi_1, \Box\Pi_1, \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Box\Pi_1 \vdash \Box(P_1 \vee P_3)}
}
$$

Execution of Leivant's algorithm translated to regular expressions

Lemma 6 *Let the region be as above, then;*

1. $\pi_{Lei}^{\varepsilon_{root}} \longmapsto \pi_{Lei}^{\varepsilon_{red}}$
2. $\pi_{Lei}^{\varepsilon_{red}} \longmapsto \pi_{Lei}^{\varepsilon_{shift}}$
3. $\pi_{Lei}^{\varepsilon_{shift}} \longmapsto \pi_{Lei}^{\varepsilon_{root}} \varrho_{Lei}$

Lemma 7 *Every proof in a reduction sequence of $\pi_{Lei_7}^{\varepsilon}$, can be written as $(\pi_{Lei}^{\varepsilon_{root}} + \pi_{Lei}^{\varepsilon_{red}} + \pi_{Lei}^{\varepsilon_{shift}})(\varrho_{Lei})^n \mu_{Lei}$.*

Lemma 8 *If we try n times to remove the cut in $\pi_{Lei_7}^{\varepsilon}$ using Leivant's algorithm then we get a new proof $\pi_{Lei}^{\varepsilon_{root}}(\varrho_{Lei})^n \mu_{Lei}$ such that $|Cut(\pi_{Lei}^{\varepsilon_{root}} \varrho_{Lei})| = 1$.*

Proof: Induction over n. Induction hypothesis gives us $\pi_{Lei}^{\varepsilon_{root}}(\varrho_{Lei})^k \mu_{Lei} \longmapsto \pi_{Lei}^{\varepsilon_{red}} \varrho_{Lei}(\varrho_{Lei})^k \mu_{Lei} \longmapsto \pi_{Lei}^{\varepsilon_{shift}} \varrho_{Lei}(\varrho_{Lei})^k \mu_{Lei} \longmapsto \pi_{Lei}^{\varepsilon_R}(\varrho_{Lei})^{k+1} \mu_{Lei}$ and we are done.

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Collaps of Leivant's reduction sequence

Lemma 9 $(\pi_{Lei}^{\varepsilon_{root}} + \pi_{Lei}^{\varepsilon_{red}} + \pi_{Lei}^{\varepsilon_{shift}})(\varrho_{Lei})^n \mu_{Lei} \longmapsto \pi_{Lei}^{\varepsilon_{root}} \varrho_{Lei} \mu_{Lei}$

Proof: By previous lemma $(\varrho_{Lei})^n \longmapsto \varrho_{Lei}$ hence $(\pi_{Lei}^{\varepsilon_{root}} + \pi_{Lei}^{\varepsilon_{red}} + \pi_{Lei}^{\varepsilon_{shift}})(\varrho_{Lei})^n \mu_{Lei} \longmapsto (\pi_{Lei}^{\varepsilon_{root}} + \pi_{Lei}^{\varepsilon_{red}} + \pi_{Lei}^{\varepsilon_{shift}}) \varrho_{Lei} \mu_{Lei}$, which gives the result.

Theorem 2 *Leivant's algorithm for cut-elimination does not terminate.*

Proof: By the previous lemmas we have shown that every reduction sequence for ε diverges, which means that we have at least one cut in each element in the reduction sequence.

Valentini's move on ε

$$
\dfrac{
\dfrac{\dfrac{\Sigma}{\vdots}\atop \Box\Pi_1 \vdash \Box(P_1 \wedge P_2) \qquad \dfrac{\Sigma}{\vdots}}{\Pi_1, \Box\Pi_1, \Box\Pi_1 \vdash P_1 \wedge P_2}\, cut_{Val}
\qquad
\dfrac{\dfrac{\Sigma}{\vdots}\atop \Box\Pi_1 \vdash \Box(P_1 \wedge P_2) \quad \dfrac{\dfrac{\dfrac{P_1 \vdash P_1}{P_1 \vdash P_1 \vee P_3}}{P_1 \wedge P_2 \vdash P_1 \vee P_3}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Box\Pi_1, P_1 \wedge P_2, \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}\, cut_2
}{
\dfrac{\dfrac{\Pi_1, \Box\Pi_1, \Box\Pi_1, \Box\Pi_1, \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}{\Pi_1, \Box\Pi_1 \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Box\Pi_1 \vdash \Box(P_1 \vee P_3)}
}\, cut_1
$$

SHIFT-arguments give π^{ε}_{Val}

$$
\dfrac{
\dfrac{\Sigma}{\vdots}\atop \Box\Pi_1 \vdash \Box(P_1 \wedge P_2)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P_1 \vdash P_1}{P_1 \wedge P_2 \vdash P_1}}{P_1 \wedge P_2, \Box(P_1 \wedge P_2), \Box P_1 \vdash P_1}}{\Box(P_1 \wedge P_2) \vdash \Box P_1}}{P_2, \Box(P_1 \wedge P_2) \vdash \Box P_1} \quad \dfrac{\dfrac{P_2 \vdash P_2}{P_2, \neg P_2 \vdash}}{\Box(P_1 \wedge P_2), P_2, \neg P_2 \vdash}}{P_2, \Box P_1 \rightarrow \neg P_2, \Box(P_1 \wedge P_2) \vdash}}{P_2, \Box P_2, \Box P_1 \rightarrow \neg P_2, \Box(\Box P_1 \rightarrow \neg P_2), \Box(P_1 \wedge P_2) \vdash}
}{
\dfrac{\dfrac{\dfrac{\Pi_1, \Box\Pi_1, \Box\Pi_1 \vdash}{\Pi_1, \Box\Pi_1, \Box\Pi_1, \Box\Pi_1, \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Pi_1, \Box\Pi_1 \Box(P_1 \vee P_3) \vdash P_1 \vee P_3}}{\Box\Pi_1 \vdash \Box(P_1 \vee P_3)}
}\, cut_{Val}
$$

Valentini's (core) algorithm on ε

Lemma 10 *If we try n times to remove the cut in* π^{ε}_{Val}*, using Valentini's algorithm with an optimal right-strategy then we get* $\pi^{\varepsilon_R}_{Val}(\varrho_{Val})^n\mu_{Val}$*, with* $|Cut(\pi^{\varepsilon_R}_{Val}\varrho_{Val})| = 1$.

Proof: Induction over n.

- Basis step: $\pi^{\varepsilon_R}_{Val}\mu_{Val} \longmapsto \pi^{\varepsilon_L}_{Val}\varrho_{Val}\mu_{Val} \longmapsto \pi^{\varepsilon_R}_{Val}\varrho_{Val}\mu_{Val}$.
- Induction-step: Ind.hyp. is just $\pi^{\varepsilon_R}_{Val}(\varrho_{Val})^k\mu_{Val}$. Then $\pi^{\varepsilon_R}_{Val}(\varrho_{Val})^k\mu_{Val} \longmapsto \pi^{\varepsilon_L}_{Val}\varrho(\varrho_{Val})^k\mu_{Val} \longmapsto \pi^{\varepsilon_R}_{Val}(\varrho_{Val})^{k+1}\mu_{Val}$ and we are done.

Church-Rosser property on the reduction sequences

The Church Rosser property says:

if $\mu \longmapsto \mu_1$ and $\mu \longmapsto \mu_2$, then there exists a $\mu_\star$ such that $\mu_1 \longmapsto \mu_\star$ and $\mu_2 \longmapsto \mu_\star$.

- The reduction sequences for ε applying Valentini's algorithm (core), form an infinite non-balanced binary tree
- In the section to come it is proven that all nodes in this tree can be reduced to one single proof still containing cut

Valentini's (core) algorithm does not terminate on ε

Lemma 11 *Every element in a reduction sequence of π^{ε}_{Val} can be written as $(\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})(\varrho^{L}_{Val} + \varrho^{R}_{Val})^n \mu_{Val}$.*

Lemma 12 *We have for the specified regions:*

1. $\pi^{\varepsilon L}_{Val} \longmapsto \pi^{\varepsilon R}_{Val}$
2. $\varrho^{L}_{Val} \longmapsto \varrho^{R}_{Val}$
3. $\pi^{\varepsilon R}_{Val} \longmapsto \pi^{\varepsilon L}_{Val}\varrho$

Lemma 13 $(\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})((\varrho^{L}_{Val} + \varrho^{R}_{Val})^n \mu_{Val} \longmapsto \pi^{\varepsilon R}_{Val}\varrho^{R}_{Val}\mu_{Val}$

Proof: Since $(\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})(\varrho^{R}_{Val} + \varrho^{R}_{Val})^n \mu_{Val} \longmapsto (\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})(\varrho^{R}_{Val})^n \mu_{Val} \longmapsto (\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})(\varrho^{R}_{Val})^n \mu_{Val} \longmapsto (\pi^{\varepsilon L}_{Val} + \pi^{\varepsilon R}_{Val})\varrho^{R}_{Val}\mu_{Val}$. But we have both $\pi^{\varepsilon L}_{Val}\varrho^{L}_{Val}\mu_{Val} \longmapsto \pi^{\varepsilon R}_{Val}\varrho^{R}_{Val}\mu_{Val}$ and $\pi^{\varepsilon R}_{Val}\varrho^{R}_{Val}\mu_{Val} \longmapsto \pi^{\varepsilon R}_{Val}\varrho^{R}_{Val}\mu_{Val}$, which proves the lemma.

Theorem 3 *Valentini's algorithm VAL-II(Core) for cut-elimination does not terminate.*

Sceleton of an algorithm

1. If there are no cuts in the tree then stop.
2. Remove all cuts were the cut formula is introduced by weakening (SHIFT).
3. Push all cuts that are propositional upwards until, they are either eliminated or they are reduced to a boxed cut.
4. Transfer all boxed cuts on the top to SNF by ROOT move.
5. All the top cuts are now boxed cuts. Make the approriate moves on the remaining top cuts according to the correct reduction move (RED).
6. Goto (1)

Concluding remarks

- A decision procedure for validity in GLS logic without cut rule: VALSAM theorem-prover
- No algorithm for eliminating cuts in GLS
- Sambin Normal Form Theorem (How many new cuts?)
- Too easy to find counterexamples to reduction moves: ε
- No real understanding of sequential calculus for provability logic
- Another rule doing better?