THE BUCKLEY-LEVERETT EQUATION WITH SPATIALLY STOCHASTIC FLUX FUNCTION

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Abstract

When the reservoir parameters are stochastic, then the flow in a reservoir is described by stochastic partial differential equations. Spatial stochastic relative permeability in one spatial dimension is modeled by the stochastic Buckley-Leverett equation $s(x,t)_t + f(s(x,t),x)_x = 0$ for $x > 0$ and $t > 0$. f is the stochastic flux function and s is the saturation. This equation is analyzed and it is proved that the solution of this equation with Riemann initial data converges to the solution of $s(x,t)_t + \bar{f}(s(x,t))_x = 0$ where $\bar{f}(s)$ is the spatial average of $f(s,x)$ when $f(s,x)$ varies randomly with position.

Key words. Buckley-Leverett equation, scalar conservation law, stochastic partial differential equations

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1. Introduction

Reservoir parameters vary spatially. The last decade it has become more usual to model this spatial variation by stochastic models see e.g. (Haldorsen & Damsleth 1990) or (Holden, Omre & Tjelmeland 1992). If the reservoir properties are modeled stochastically, the reservoir simulation becomes a numerical solution of a stochastic differential equation. This is usually solved by intensive use of computer resources: generate a realization of the reservoir and solve the differential equation with the input data from the realization. The solution of the stochastic differential equation is found from the statistics of the solutions of the differential equation. This approach is f.ex. used by (Omre, Tjelmeland, Qi & Hinderaker 1991) and (Langtangen 1988).

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The theory of stochastic partial differential equations has successfully been used to analyze many problems of applied mathematics. However, these equations are mostly linear or first-order with randomness expressed in terms of "white noise" i.e. the derivative of Brownian motion (Øksendal 1992) which makes it hard to apply to reservoir simulation.

In this paper it is made stronger assumptions on the equations modeling the flow and hence it is possible to do a rigorous analysis of a nonlinear equation with more complicated stochastic properties. We will study the Buckley-Leverett equation which models incompressible, immiscible two-phase flow in a porous medium in one spatial dimension. Stochastic Buckley-Leverett equation has previously been studied (Holden & Risebro 1991). They found the solution when the flux function was stochastic but not varying in space. In this paper the flux function varies spatially. In real reservoirs there is a very large spatial variability. In the Buckley-Leverett equation it is trivial to handle the case of spatial variable permeability. The practical application of this result is therefore to handle spatial varying relative permeability. We will assume that the flux function is monotonic. In (Langtangen, Tveito & Winther 1992) it is shown that the Buckley-Leverett is unstable if the flux function is not monotonic.

Geologist usually model the reservoir in a much finer detail than it is possible to put into a reservoir simulator, see e.g. (Haldorsen & Damsleth 1990). It is necessary to find effective values for the parameters in larger blocks which can be put into a reservoir simulator. There is a large number of papers on finding effective permeabilities. Most of these techniques are ad hoc see e.g. (King 1989). There are also some papers on effective relative permeabilities, see (Ekrann & Dale 1992). Recently (Tjølsen, Damsleth & Bu 1993) have by intensive use of reservoir simulation shown that a spatial varying relative permeability can be replaced by the average relative permeability without changing the reservoir performance considerably. This paper will confirm their conclusion by a rigorous solution of the displacement in one dimension.

2. The Buckley-Leverett equation

In this paper we will model displacement of two phases in one spatial dimension neglecting gravity, compressibility and capillarity using the standard equations used in reservoir simulation. The velocity of a phase i, v_i is modeled by Darcy's Law:

$$v_i = -\frac{k(x)}{\mu_i} k_{r,i}(s_i, x) p_x(x) \qquad \text{for } i = w, o$$

where $k(x)$ is the permeability, μ_i is the viscosity of phase i, $k_{r,i}(s_i, x)$ is the relative permeability of phase i with saturation s_i, p is the pressure and x the spatial variable. The indexes w and o stand for water and oil. Conservation of phase i gives:

$$\phi s_{i_t} + v_{i_x} = 0 \qquad \text{for } i = w, o.$$

where ϕ is the porosity. Adding the equation for conservation of each phase together using that $s_o + s_w = 1$ gives:

$$(k(x)(\frac{k_{r,o}(s_o, x)}{\mu_o} + \frac{k_{r,w}(s_w, x)}{\mu_w})p_x(x))_x = 0.$$

The pressure is then

$$p(x) = p_0 + \int_0^x \frac{a}{k(x)(\frac{k_{r,o}(s_o,x)}{\mu_o} + \frac{k_{r,w}(s_w,x)}{\mu_w})}dx$$

where the constants a and p_0 are determined by the boundary condition. This is put into the equation for conservation of the water phase:

$$\phi s_t + (v_w + v_o)f(s, x)_x = 0$$

where the flux function is

$$f(s, x) = \frac{v_w}{v_w + v_o} = \frac{\frac{k_{r,w}(s,x)}{\mu_w}}{\frac{k_{r,w}(s,x)}{\mu_w} + \frac{k_{r,o}(1-s,x)}{\mu_o}}$$

where the w index is neglected in the saturation s. The saturation of the oil is found from $s_o = 1 - s_w$. For the rest of the article we will assume that $\frac{v_w + v_o}{\phi}$ is constant equal to one. This is only a scaling of the flux function f. This is the well-known Buckley-Leverett equation if we neglect the spatial variability (Peaceman 1977). It is usual to assume that $k_{r,w}(s_w, x)$ and $k_{r,o}(s_o, x)$ have the form shown in Figure 1. This gives the flux function the typical s-shape shown in Figure 2.

We will study the solution for $x > 0$ with the boundary conditions $f(s_w(x, 0), x) = c_1 \geq 0$ for $x > 0$ and $f(s_w(0, t), 0) = c_2 > c_1$ and assume that the flux function is increasing in s. This is a typical water flooding situation where the reservoir is filled by oil $s_o(x, 0) = 1 - s_w(x, 0)$ and water is flooding in from $x = 0$. The solution of the Buckley-Leverett equation shows the displacement of oil by water.

In (Oleinik 1963) it is proved that there is a unique solution of the equation also with a flux function depending in the spatial variable when the flux function is continuous in x. Uniqueness for discontinuous flux function is proved in (Holden & Høegh-Krohn 1990) and in (Gimse & Risebro 1990).

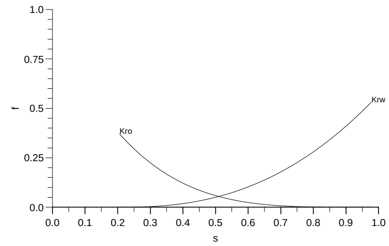

FIGURE 1. Relative permeabilty for water and oil as a function of water saturation. The relative permeabilities are only shown in the mobile interval, e.g. $S_{cw} < S < 1 - S_{ro}$

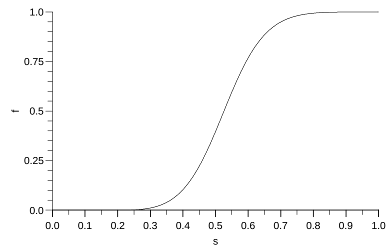

FIGURE 2. Fractional flow of water as a function of water saturation.

3. Piecewise linear flux function

First we will assume that the flux function is piecewise constant in x and continuous and piecewise linear in s and the initial saturation distribution is piecewise constant. This gives a particular simple solution (Dafermos 1972).

In a later section this discretisation is refined and the results are proved for more general flux functions. This approach was used in (Holden & Holden 1992) which was inspired by previous work (Glimm 1965). This is formalized as follows:

Divide the spatial distribution in the intervals

$$0 = x_0 < x_1 < \ldots$$

The flux function may be written

$$f(s,x) = f_{j-1}\frac{s_{i,j} - s}{s_{i,j} - s_{i,j-1}} + f_j\frac{s - s_{i,j-1}}{s_{i,j} - s_{i,j-1}} \quad \text{for } s_{i,j-1} < s < s_{i,j} \text{ and } x_{i-1} \le x < x_i.$$

See Figure 3. The solution depends on the concave envelope of the flux function $f_c(s)$, see Figure 4. Notice that the concave envelope is defined relative to the end-points s_- and s_+. This is not included in the notation, except in the cases where

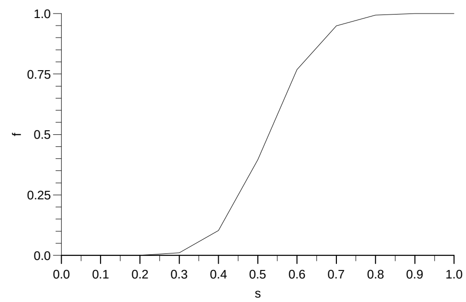

FIGURE 3. Fractional flow of water approximated by a piecewise linear function.

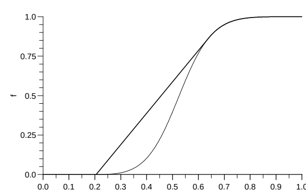

FIGURE 4. The fractional flow curve of water and its concave envelope as a function of water saturation.

it may be confusing. Then the notation $f_{c(s_-,s_+)}(s)$ is used. The solution contains discontinuities. These discontinuities are called shocks.

Assume first there is only one interval, i.e.

$$f(s,x)=\begin{cases} f_-(s) & \text{for } x\le 0\\ f_+(s) & \text{for } x>0.\end{cases}$$

Let the initial situation be

$$s(x,0)=\begin{cases} s_- & \text{for } x\le 0\\ s_+ & \text{for } x>0.\end{cases}$$

with $f_-(s_-) > f_+(s_+)$. Since the flux function is increasing in s, the solution is constant for x negative.

The solution is (see e.g (Holden & Høegh-Krohn 1990) or (Gimse & Risebro 1990).)

$$s(x,t)=\begin{cases} s_- & \text{for } x\le 0\\ s_{1,m+1} & \text{for } 0<\frac{x}{t}<v_m\\ s_{1,j} & \text{for } v_j<\frac{x}{t}<v_{j-1} \text{ for j=1,...,m}\\ s_+ & \text{for } v_0<\frac{x}{t}\end{cases}$$

where the velocities are defined by

$$v_j=\frac{f_j-f_{j-1}}{s_{1,j}-s_{1,j-1}}.$$

f_j are the breakpoints of the concave envelope of the flux function i.e. $f_j=f_{+c}(s_{1,j})$. $s_{1,0}$ and $s_{1,m+1}$ are defined by $s_{1,0}=s_+$ and $f_+(s_{1,m+1})=f_-(s_-)$ respectively.

If $f_-(s_-)<f_+(s_+)$, the solution is as described above but with x and f replaced by $-x$ and $-f$ respectively. If f is not monotone in s, the solution is more complicated, see (Holden & Høegh-Krohn 1990).

If f is independent of x, $s(x,t)$ is monotone in x for a fixed t. But when f depends on x, then $s(x,t)$ is not necessarily monotone since we may have $s_{1,m+1}>s_-$. The function $f(s(x,t),x)$ is however monotone decreasing since f is monotone in s, and where $s(x,t)$ may increase $f(s(x,t),x)$ is constant since $f_+(s_{1,m+1})=f_-(s_-)$.

If there are several x-intervals, the solution is found by the following construction: *Follow all the shocks from the initial discontinuity in the saturation. Each time two shocks collide or a shock collide with a x-discontinuity in the flux function, solve the Riemann problem as above.*

A solution may be as shown in Figure 5. This approach was used in (Holden & Holden 1992).

In the first proposition we will need the following definitions:

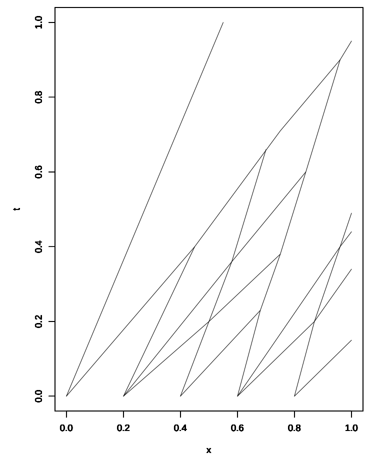

FIGURE 5. A typical solution is piecewise constant in the x-t plane.

The spatial average of $f(s,x)$ for $0 < x < \bar{x}$, $\bar{f}(s)$ is defined by

(1) *Define* $g_1(y,x)$ *as the* s *inverse of* f *by* $g_1(f(s,x),x) = s$.
(2) *Define* $g_2(y)$ *as the spatial average of* $g_1(y,x)$ *by* $g_2(y) = \frac{\int_0^{\bar{x}} g_1(y,x)dx}{\bar{x}}$.
(3) *Define* $\bar{f}(s)$ *as the inverse of* $g_2(y)$ *by* $\bar{f}(g_2(y)) = y$.

$\bar{f}_c$ is defined similarly from the concave envelope f_c of f.

Property A relative to the interval (c_1, c_2) is defined as:
There is a constant d *such that* $f_c(s,x) = f(s,x)$ *for* $f(s,x) > d$ *and* $f_c(s,x) > f(s,x)$ *for* $f(s,x) < d$ *where* f_c *is the concave envelope of* f *relative to the constants* c_1 *and* c_2.

In the flooding problem property A implies that the flux function has the same value behind the shock for all the x-intervals. Property A is satisfied if f is concave by $d = f(c_1, x)$ for all $x > 0$ and if f is convex by $d = f(c_2, x)$ for all $x > 0$.

We may then state the following proposition for a typical water flooding situation :

Proposition 1

Assume the flux function $f(s,x)$ *is piecewise constant in* x *and continuous, increasing and piecewise linear in* s.

Then the solution $s(x,t)$ *of*

$$s_t + f(s,x)_x = 0$$

for $x > 0$ with the boundary condition $f(s(x,0),x) = c_1$ for $x > 0$ and $f(s(0,t),0) = c_2 > c_1$ satisfies the following:

1. *$s(x,t)$ is piecewise constant,*
2. *$f(s(x,t),x)$ is piecewise constant in (x,t) and decreasing in x for fixed $t > 0$,*
3. *for each fixed value of x, $\bar{x} > 0$, there exists a function $h(s)$ satisfying*

$$\bar{f}(s) \leq h(s) \leq \bar{f}_c(s)$$

such that the value of the flux for the solution of

$$s_t + f(s,x)_x = 0$$

and

$$s_t + h(s)_x = 0$$

are identical for $x = \bar{x}$. If $f(s,x)$ satisfies property A, then $h_c(s) = \bar{f}_c(s)$.

It is trivial to prove that there exists an effective flux function $h(s)$. The main result is that the flux function is bounded by $\bar{f}(s)$ and $\bar{f}_c(s)$. If property A is satisfied, the effective flux function is equal to $\bar{f}(s)$. If property A is not satisfied, the effective flux function depends on $f(s,x)$ in each x intervall. The function $h(s)$ is not uniquely defined since only the concave envelope of $h(s)$ defines the solution. The concave envelope of $h(s)$ is however uniquely defined or equivalent there is a unique concave $h(s)$.

The solution of the saturation at $x = \bar{x}$, $s(\bar{x},t)$ is found from the flux function $f(s,\bar{x})$.

Proof of proposition 1

The solution $s(x,t)$ is found by the construction of a series of solutions of Riemann problems as described in the beginning of this section. The same technique is used in (Holden & Holden 1992).

It follows directly from this construction that $s(x,t)$ and $f(s(x,t),x)$ are piecewise constant and $f(s(x,t),x)$ is decreasing in x for a fixed t value. $f(s(x,t),x)$ is continuous over the lines $x = x_i$ except when a shock intersects the line $x = x_i$.

It is left to show that the flux $f(s(x,t),x)$ for a fixed $x = \bar{x}$ is the value of the flux for the solution where $f(s,x)$ is replaced by a function $h(s)$. We may assume that $x = \bar{x}$ is one of the discontinuity points of $f(s,x)$

It follows from the construction that $s(x,t)$ only takes the values $s_{i,j}$ where there is a break point in $f(s,x)$. Similarily, $f(s(x,t),x)$ only takes the values f_j. We assume that there is a break point in $f(s,x)$ for $s = c_1$ and for $f(s,x) = c_2$ in each x-interval.

Denote the shock in $f(s(x,t),x)$ from the value f_j to f_{j-k} for f_j^{j-k}. $k=1$ except for the case where there is a shock over an interval where $f_c(s,x) > f(s,x)$.

The solution is most easily described by finding the velocity of each shock, f_j^{j-k}.

Assume first that f satisfies property A. Then no shocks will collide or be split in several smaller shocks. The velocity of a shock f_j^{j-k} in the interval $x_{i-1} < x < x_i$ is $v_{i,j} = \frac{f_j - f_{j-k}}{s_{i,j} - s_{i,j-k}}$.

Let $s_j = \sum_{i=1}^n \frac{x_i - x_{i-1}}{x_n} s_{i,j}$ where $x_n = \bar{x}$. Then $\bar{f}(s_j) = f_j$. Let $t_{i,j}$ be the time the f_j^{j-k} shock uses to pass interval (x_{i-1}, x_i). The average velocity of the f_j^{j-k} shock in $(0, \bar{x})$ is:

$$u_j^{j-k} = \frac{\bar{x}}{\sum_{i=1}^n t_{i,j}} = \frac{\bar{x}}{\sum_{i=1}^n \frac{x_i - x_{i-1}}{v_{i,j}}} = \frac{\bar{x}}{\sum_{i=1}^n \frac{(x_i - x_{i-1})(s_{i,j} - s_{i,j-k})}{f_j - f_{j-k}}} =$$
$$\frac{f_j - f_{j-k}}{\sum_{i=1}^n \frac{x_i - x_{i-1}}{x_n}(s_{i,j} - s_{i,j-k})} = \frac{f_j - f_{j-k}}{s_j - s_{j-k}} = \frac{\bar{f}(s_j) - \bar{f}(s_{j-k})}{s_j - s_{j-k}}$$

which is the speed if $f(s,x)$ is replaced by $\bar{f}(s)$.

It is left to prove that also in the case that assumption A is not satisfied, it is possible to replace $f(s,x)$ by $h(s)$ where $\bar{f}(s) \le h(s) \le \bar{f}_c(s)$. It is then necessary to handle the situation of collision of shocks and that shocks split in several smaller shocks.

The shocks f_j^{j-k} and $f_{j-k}^{j-k'}$ collide when their average velocities u_j^{j-k} and $u_{j-k}^{j-k'}$ becomes equal. This is exactly the situation where the concave envelope of $\bar{f}(s)$ in $(s_{j-k'}, s_j)$ becomes different from $\bar{f}(s)$. The solution of $s_t + \bar{f}_x = 0$ changes character from separate shocks f_j^{j-k} and $f_{j-k}^{j-k'}$ to one single large shock $f_j^{j-k'}$. Therefore also in this situation we get exact solution with $h(s) = \bar{f}(s)$ replacing $f(s,x)$ as long as shocks do not splitt.

In the more complicated situation where shocks split up it is necessary to define $h(s)$ from the solution $s(x,t)$. If a shock $f_j^{j-k'}$ splits up in two smaller shocks f_j^{j-k} and $f_{j-k}^{j-k'}$, this may be handled as if the two shocks existed already, but were overlapping. This would be the case if the flux function $f(s,x)$ was replaced by its concave envelope $f_c(s,x)$ in the intervals where the shocks were overlapping. Therefore it is possible to replace $f(s,x)$ with $h(s) = \bar{f}_d(s)$ where $f_d(s)$ is defined by

$$f_d(s,x) = \begin{cases} f_{c(s_{i,j-k}, s_{i,j})}(s,x) & \text{for } x' < x < x'' \text{ and } s \text{ such that shocks } f_j^{j-1}, \ldots, f_{j-k+1}^{j-k} \\ & \text{are overlapping in the interval } x' < x < x'' \\ f(s,x) & \text{else.} \end{cases}$$

This covers the situation where there are both splittings and collisions. As stated earlier the effective flux function is not uniquely defined since only the concave envelope of the flux function defines the solution. The above definition makes $h(s)$ concave and as large as possible. Since $f(s,x) \leq f_d(s,x) \leq f_c(s,x)$, we have that $\bar{f}(s) \leq h(s) = \bar{f}_d(s) \leq \bar{f}_c(s)$. ■

This proposition describes the solution $s(x,t)$ when the spatial variability in the flux function is removed and the flux function is replaced by an average flux function $h(s)$ which satisfies $\bar{f}(s) \leq h(s) \leq \bar{f}_c(s)$. The solution does not only depend on an average of the flux function, but also for which x values the function $f(.,x)$ takes the different values. In the following section we will prove that if the variability in $f(.,x)$ is random, it is possible to give a stronger result.

4. Stochastic flux function

Also in this section we will assume that the flux function is piecewise constant in x and continuous and piecewise linear in s. In addition, it is assumed in this section that the flux function $f(s,x)$ is stochastic as a function of x i.e. that there is a distribution for the function $f(.,x)$. We will assume that this distribution is independent of x and there are no spatial correlation. Then the stochastic functions $\{f(s,x_0), f(s,x_1), f(s,x_2), ...\}$ are identical distributed independent stochastic functions. $x_0, x_1, ...,$ are the endpoints of the intervals where $f(s,x)$ is constant as a function of x.

We may formulate the following proposition:

Proposition 2

Assume the flux function $f(s,x)$ satisfies

(1) *Piecewise constant in x and the intervals where it is constant, is of equal length.*
(2) *Continuous, increasing and piecewise linear in s.*
(3) *In each x interval $f(s,x)$ is a stochastic function of s and independent and identical distributed for each x interval.*

Let $f_n(s,x)$ satisfy the above requirements where n is the number of x interval where $f_n(s,x)$ is constant in $(0,\bar{x})$. The solution $s_n(x,t)$ of

$$s_t + f_n(s,x)_x = 0$$

for $x > 0$ with the boundary condition $f(s(x,0),x) = c_1$ for $x > 0$ and $f(s(0,t),0) = c_2 > c_1$ satisfies the following: $f_n(s_n(\bar{x},t),\bar{x}) \to \bar{f}(\bar{s}(\bar{x},t))$ pointwise in t when $n \to \infty$

with probability 1. $\bar{s}(x,t)$ *is the solution of*

$$s_t + \bar{f}(s)_x = 0.$$

$\bar{f}(s)$ *is defined from the distribution of* $f(\cdot, x)$

In this proposition we state that when the number of intervals n increases and $f(s,x)$ is independent and identical distributed in each x interval, $f_n(s_n(x,t),\bar{x}) \to \bar{f}(\bar{s}(x,t))$. The effective flux function $h(s)$ is not uniquely defined since only the concave envelope of the flux function defines the solution. It is however independent of $\bar{x}$.

Proof of proposition 2

For a given function $f_n(s,x)$ we have from proposition 1 that $h_{n,c}(s(\bar{x},t)) = f_{n,c}(s(\bar{x},t),\bar{x})$ and

$$\bar{f}_n(s) \leq h_n(s) \leq \bar{f}_{n,c}(s).$$

It is obvious that $\lim_{n\to\infty} \bar{f}_n(s) = \bar{f}(s)$ with probability 1 from the law of large numbers where $\bar{f}(s)$ is defined from the distribution of $f(s,x)$. It is left to prove that $\lim_{n\to\infty}(\bar{f}_{n,c}(s) - h_{n,c}(s)) = 0$ i.e. that the effective flux function approaches the spatial average flux function.

Assume that several neighboring shocks are not overlapping in the solution of $\bar{s}(x,t)$. The expected velocity for these shocks are increasing with decreasing values of s. Therefore, the probability for an overlap at $x = \bar{x}$ will vanish as $n \to \infty$. These shocks may have a different velocity than the slope of $\bar{f}_n(s)$ due to an overlap in some (early) interval. But the difference in average velocity will vanish as $n \to \infty$.

Assume the contrary, that several shocks are overlapping in the solution $\bar{s}(x,t)$, i.e. $(\bar{f})_c(s,x) \geq \bar{f}(s,x)$ for $d_1 < f(s,x) < d_2$. By the law of large numbers there will only be interaction for small values of x when n is large. Hence, it is possible to study a sequence of shocks which overlap in the solution $\bar{s}$ isolated from other shocks. Part of the time shocks which overlap in the solution of $\bar{s}$, will be overlapping and part of the time not overlapping in the solution of $s_n(x,t)$. These shocks split up in some of the x intervals if $f_c(s,x) = f(s,x)$. But since the expected average velocity is larger for larger values of s these shocks will for n sufficent large be overlapping arbitrary close to $\bar{x}$ with probability 1. When the shocks join to one large shock, this shock will have exactly the same velocity as the slope of $\bar{f}_n(s)$ since the velocity only depends on the values e_i where $h_n(e_i) = d_i$ for i=1,2. The velocity with spatial variable flux is therefore arbitrary close to the velocity when the flux function is $\bar{f}$ ■

5. General flux function

In the previous sections we had very strong assumptions on the flux function, piecewise constant in x and piecewise linear in s. It is possible to approximate a more general flux function with functions satisfying these strong assumptions. In this section we will prove similar theorems by approximating the flux function by flux functions which satisfy the assumptions in the propositions in the previous sections.

Theorem 1

The solution $s(x,t)$ of

$$s_t + f(s,x)_x = 0$$

for $x > 0$ with the boundary condition $f(s(x,0),x) = c_1$ for $x > 0$ and $f(s(0,t),0) = c_2 > c_1$ and $f(s,x)$ piecewise continuous in x and continuous and increasing in s for fixed x satisfies the following:

(1) *$f(s(x,t),x)$ is decreasing in x for fixed $t > 0$,*

(2) *for a fixed $\bar{x} > 0$, the flux of the solution is identical to the flux of the solution of the equation*

$$s_t + h(s)_x = 0$$

where $h(s)$ satisfies

$$\bar{f}(s) \le h(s) \le \bar{f}_c(s).$$

If f satisfies property A, then

$$h_c(s) = \bar{f}_c(s).$$

Proof of theorem 1

Define a sequence of flux functions $f_i(s,x)$ for i=1,2,... which satisfies:

(1) The assumptions in proposition 1.

(2) $f_i(s,x) \to f(s,x)$ and $\frac{\partial f_{i+1}(s,x)}{\partial s} \ge \frac{\partial f_i(s,x)}{\partial s}$ for $f_i(s,x) > c_1$.

This is f.ex. satisfied by letting $f_i(s,x)$ be the maximum function which satisfies requirements above and which have breakpoints in $(s,x) = (\frac{n}{2^i}, \frac{m}{2^i})$ for integers n and m. Let the influx boundary condition be an increasing sequence $c_1 < f_i(s(0,t),0) \le c_2$ while the initial condition is constant $f_i(s(x,0),x) = c_1 < c_2$. Then the corresponding solutions $s_i(x,t)$ are an increasing sequence for each fixed (x,t) since $\frac{\partial f_i(s,x)}{\partial s}$ increases when i increases.

Since the solutions are bounded by s such that $f(s,x) = c_2$, the sequence $s_i(x,t)$ will converge for each (x,t). Define $s(x,t) = \lim_{i\to\infty} s_i(x,t)$

First we prove that $s(x,t)$ is a solution of the equation. Let $\phi(x,t)$ be a continuous differential function with bounded support in the interior of $x>0$ and $t>0$. $s_i(x,t)$ is a weak solution, i.e.

$$0=\int\int(s_{i,t}+f_{i,x}(s_i,x))\phi dxds=-\int\int(s_i\phi_t+f_i(s_i,x)\phi_x)dxds.$$

Since $s(x,t)=\lim_{i\to\infty}s_i(x,t)$,

$$\int\int(s\phi_t+f(s,x)\phi_x)dxds=0.$$

Therefore, $s(x,t)$ is a solution of the equation.

Since $f_i(s(x,t),x)$ is decreasing in x for fixed $t>0$ also $f(s(x,t),x)$ is decreasing in x for fixed $t>0$.

The effective flux function $h(s)$ may be defined as $h(s)=\lim_{i\to\infty}h_i(s)$. This sequence converges since it is increasing and bounded. Obviously the inequalities

$$\bar{f}(s)\le h(s)\le\bar{f}_c(s)$$

also holds in the limit and the solution is uniquely defined if assumption A is satisfied. ■

It is also possible to generalize proposition 2:

Theorem 2

Consider the solution $s(x,t)$ of

$$s_t+f_x(s,x)=0$$

for $x>0$ with the boundary condition $f(s(x,0),x)=c_1$ for $x>0$ and $f(s(0,t),0)=c_2>c_1$ and where $f(s,x)$ is piecewise constant in x, constant in interval of equal length, continuous and increasing in s, $f(0,x)=0$ and $f(1,x)=1$. Assume in addition that $f(s,x)$ is stochastic as a function of x and for each x interval $f(s,x)$ are independent and identical distributed. Then the solution $s_n(x,t)$ where n is the number of x intervals in $(0,\bar{x})$, satisfies $f_n(s_n(\bar{x},t))\to\bar{f}(\bar{s}(\bar{x},t))$ pointwise in t with probability 1 when $n\to\infty$. $\bar{s}(x,t)$ is the solution of

$$s_t+\bar{f}(s,x)_x=0.$$

This theorem is proved similarly as the proof of theorem 1 using proposition 2.

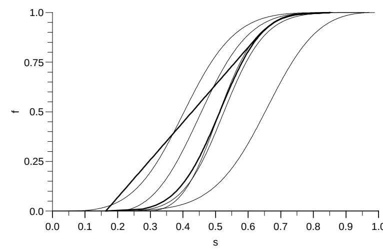

FIGURE 6. Five typical fractional flow curves and the averages $\bar{f}(s)$ and $\bar{f}_c(s)$ calculated from the distributions. Notice that $\bar{f}_c(s)$ is almost equal to the concave envelope of $\bar{f}(s)$. This implies that for this data set it is no practical importance wether the average is taken before or after the concave envelope operation.

6. Numerical example

The theorems in the previous section are illustrated by a numercial example. Let the relative permeability functions be

$$k_{rw}(S) = \begin{cases} 0 & S < S_{cw} \\ (S - S_{cw})^{\alpha} & S > S_{cw} \end{cases}$$

$$k_{ro}(S) = \begin{cases} 0 & S > 1 - S_{ro} \\ (1 - S_{ro} - S)^{\beta} & S < 1 - S_{ro} \end{cases}$$

and the fractional flow function be

$$f(S) = \frac{\frac{k_{rw}(S)}{\mu_w}}{\frac{k_{ro}(S)}{\mu_o} + \frac{k_{rw}(S)}{\mu_w}}$$

where

$$\begin{aligned} S_{cw} &\sim \text{Uniform}(0, 0.3) \\ S_{ro} &\sim \text{Uniform}(0, 0.3) \\ \alpha &\sim \text{Uniform}(1.5, 4) \\ \beta &\sim \text{Uniform}(1.5, 4) \\ \mu_w &\sim \text{Uniform}(1, 1.5) \\ \mu_o &= 1.0 \end{aligned}$$

Figure 1 and Figure 2 show typical relative permeability and fractional flow functions from this distribution. Figure 6 shows five different relative permeability functions from this distribution. In addition it is shown the spatial average flux function $\bar{f}(s)$ and the spatial average of the convex envelopes, $\bar{f}_c(s)$. Notice that $\bar{f}_c(s)$ is almost equal to the concave envelope of $\bar{f}(s)$. This implies that for this data set it is no practical importance wether the average is taken before or after the concave envelope

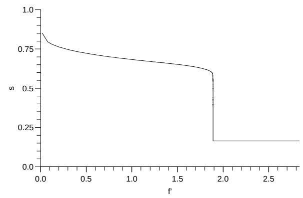

FIGURE 7. The solution $s(x,t)$ for a fixed value of t with the effective flux function as the flux function.

operation. The size of the interval for the effective flux functions in theorem 1 is in practise neglectable.

Figure 7 shows the solution $s(x,t)$ for a fixed value of t for the above problem with the effective flux function as the flux function.

Notice that the endpoints of $\bar{f}(s)$ and $\bar{f}_c(s)$ are both equal the arithmetic average of the endpoints in each interval if ϕ is constant. This implies that the effective endpoints are equal to the arithmetic average of the endpoints in each interval if ϕ is constant. Therefore, we may expect that $(\bar{f})_c(s)$ and $\bar{f}_c(s)$ are almost equal for quite general distributions.

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