Norsk Regnesentral
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Note

A Summary of Real Options Methodology

Note no SAMBA/31/05
Author Nils F. Haavardsson

Date September 30, 2005

Norwegian Computing Center

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Norsk Regnesentral
Norwegian Computing Center
Postboks 114, Blindern
NO-0314 Oslo, Norway

Besøksadresse
Office address
Gaustadalléen 23
NO-0373 Oslo, Norway

Telefon · telephone
(+47) 22 85 25 00
Telefaks · telefax
(+47) 22 69 76 60

Internett · internet
www.nr.no
E-post · e-mail
nr@nr.no

Title	**A Summary of Real Options Methodology**
Author	**Nils F. Haavardsson** <nilsfri@math.uio.no>
Date	September 30, 2005
Publication number	SAMBA/31/05

Abstract

This note presents a non-comprehensive summary of the contributions made in the real options literature.

Keywords	Real options, discounted cash flow method, contingent claims analysis, dynamic programming, hybrid techniques
Target group	Project managers
Availability	Open
Project	Statistical Analysis of Risk
Project number	830105
Research field	Technology, industry and administration
Number of pages	38

Contents

1 Introduction

When oil companies face development prospects of an oil field, they typically base their decisions on the expected Net Present Value (NPV) of the project. In order to obtain a good estimate of this number many sources of uncertainty should be taken into account. The most important contributors to the uncertainty are:

- Uncertainty about the available amount of resources, i.e. how much oil or gas can be produced from the field.
- Uncertainty about the oil and gas price, i.e. how much profit can be made from production.
- Uncertainty about the investment cost (CAPEX), i.e. how much will we have to invest, mainly before production can begin (facilities, drilling).
- Uncertainty about the operating costs (OPEX), i.e. how much will it cost to run the field during the production phases.

Traditional valuing techniques like the discounted cash flow method or Net Present Value (NPV) calculations, possess several weaknesses that make them suboptimal for valuing an oil field development project. The sources of uncertainty listed above can not be incorporated adequately using these valuation models. Another important weakness is related to management's ability to respond to the evolution of these uncertainties through decisions. The traditional valuation techniques are not able to take into account how these decisions affect the project value. Management may for example postpone a project if for example oil prices are particularly low. During a project it may be of interest to temporarily halt the project when needed while waiting for conditions to improve. Operational decisions such as platform capacity, purchase of processing equipment and tuning of the number of production wells are also difficult to incorporate using traditional techniques. Management may also at any time, at least in theory, terminate the project if it considers that to be optimal. This managerial flexibility represents options that the company holds. In contrast to financial options they can not be traded on a stock exchange. Hence they are called real options.

During the past decades there have been several attempts to develop mathematical, statistical and economic models to value real options. Early contributions were to a large extent analytical in nature. These contribution utilize the framework of option pricing theory introduced by Black and Scholes (1973). We begin our summary by highlighting the most important features of the analytical

approaches. It turned out to be hard to find explicit solutions to the problems stated in these analytical approaches. As a consequence academics and practitioners were forced to simplify the models to a level at which they lost relevance to real-world decision makers. As a response to this development, numerical approaches became increasingly popular and in demand. The evolution of the computer and software industry also played an important part in this development. In this summary we will treat the two most important and relevant classes of numerical approaches that have evolved in the real options literature. The two classes treated are:

- Stochastic programming. Two techniques will be treated, namely stochastic dynamic programming and scenario aggregation.
- Hybrid techniques. Five techniques will be treated; random tree methods, stochastic mesh methods, regression methods, parametric methods and state-space partition methods.

2 Analytical approaches

2.1 General framework

Option pricing theory and contingent claims analysis offers an efficient framework for the valuation of corporate assets and liabilities. Although the option pricing models developed by Black and Scholes (1973) are founded on many simplifying and unrealistic assumptions, option pricing theory has become increasingly popular after these papers were published. Black and Scholes designed their option pricing models for the valuation of tradable assets. The real option community has applied the framework of Black and Scholes on real investments treating them as real assets, although these assets are not tradable.

Black and Scholes (1973) constructed a portfolio of tradable assets that replicated the risk profile of the instrument of interest. In this way they constructed a portfolio that was riskless for a short period of time. Dixit and Pindyck (1994) adopted this framework and assumed the existence of spanning assets; i.e. they assumed that there exits a "complete" market where all project cash flows may be replicated by trading securities in the market. More formally, the securities market is complete if, for every project c there exists a $replicating\ strategy\ \beta$, that generates cash flows which exactly match the project's future cash flows at all times and in all states. If this assumption is realistic, it is thus possible to hedge all project cash flows for a short period of time by purchasing tradable assets in the market.

Then the portfolio of interest also becomes risk-less for a short period of time, analogous to the risk-less portfolio proposed by Black and Scholes.

The crucial question then becomes whether such spanning assets can be found or not. The answer to this question is closely related to the level of analysis. Boehren and Ekern (1985) presents six levels for the analysis of an oil field development project. Figure 1 illustrates the investment analysis hierarchy in the article of Boehren and Ekern (1985). The arrow in figure 1 symbolizes the in-

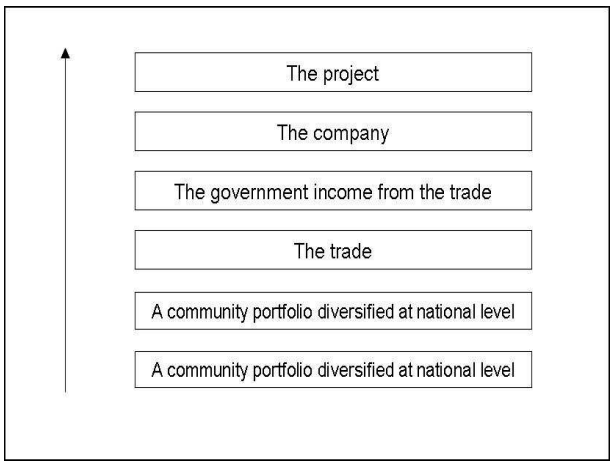

Figure 1. Investment analysis levels as presented in Boehren and Ekern (1985)

creasing multitude of non-diversifiable risks at the given analysis level. Thus, project level possesses the greatest multitude of non-diversifiable risks, while national community level contains the second smallest amount of non-diversifiable risks in this framework. At project level, the uncertainty associated with the reservoir volume, investment costs and maintenance costs represent risky cash flows that cannot be mirrored by tradable assets in the market. At national community level, however, it may be reasonable to assume that a well diversified portfolio of projects and companies and trades only contains market risk. Since we are mainly concerned with the project level, the assumption of spanning assets is not realistic for our purpose. Option theory in its purest form will thus not suffice to model the real options realistically since reservoir uncertainty is such an important element in our analysis.

2.2 Previous contributions

Many contributions have focused on only one source of uncertainty; typically uncertainty in the price of oil or gas. The price uncertainty is often modelled using a Geometric Brownian Motion (GMB). In the early contributions the price is often modelled using a one-factor model. In later contributions these one-factor model are typically refined to more complex two- and three-factor models. The

number of factors here refers to the dimension of the response variable vector. A one-factor model typically models only the spot price; i.e. the dimension of the response variable vector is 1. A two-factor model typically models the spot and for example the convenience yield; i.e. the dimension of the response variable vector is 2.

The reservoir volume is often assumed to be deterministic. In some cases a very simple reservoir model is applied. In application 2.3.1 below we see an example of such a simple model; reservoir depletion is here assumed to follow an exponential decline. Some contributions model the reservoir as a Geometric Brownian Motion, see i.e. Pindyck (1980) or Ekern and Stensland (1993). This is not an adequate model for many reasons. The most important reason is that, under GMB assumptions, the variable changes value by the simple passage of time, whereas in most real life application our knowledge of the reservoir changes only by exercising a learning option. Increased knowledge of important reservoir parameters such as porosity and permeability is gained by exercising learning options such as the drilling of exploration wells or shooting new seismic. The technical uncertainty represented in the reservoir volume uncertainty is thus correctly indexed by events, not by time units as in GMB and many other stochastic processes.

2.3 Applications of analytical approaches

A number of applications of analytical approaches are available. Our main focus is to model risk at project level. Bearing that in mind, the contributions in the literature most relevant will be summarized below. Other contributions related to oil field development projects are Bjerksund and Ekern (1990), Ekern (1988), McDonald and Siegel (1986), Pickles and Smith (1993) and Pindyck (1980).

2.3.1 Evaluating natural resource investments

Brennan and Schwartz (1985) present one of the earliest contributions in evaluating natural resource investments. In their paper they develop a general one-factor model for valuating a copper mine. Making some simplifying assumptions, a specialized version of the general model is presented. This specialized version contains a closed-form solution of the valuation problem. A numerical example based on the general model is also presented. The finite difference method is applied in the numerical example. For a thorough explanation of the method of finite differences, see Jonsbraaten (1998). Further, the optimal timing of this natural resource investment is discussed. Finally, the problems of pricing fixed price long term contracts using this methodology are briefly discussed.

2.3.2 Option valuation of claims on real assets. The case of offshore petroleum leases

Paddock et al. (1988) model an offshore petroleum lease as a three stage process where exploration, development and extraction represent the stages. If the exploration results are promising, the firm may proceed to the development stage, where equipment such as platforms and drilling wells is put in place to extract the oil. In this stage an $undeveloped$ reserve is turned into a $developed$ reserve. These three stages consequentially form a nested sets of options; the offshore petroleum lease is thus valued as a compound option.

The extraction stage is valued first. Assuming that there exists a petroleum reserve market equilibrium, the value of a producing developed reserve is modelled as a Geometric Brownian Motion

$$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v,$$

where α_v is the expected rate of capital gain, σ_v is the instantaneous standard deviation of the rate return per unit time and dz_v is an increment to a Wiener process. The production from a developed reserve is assumed to follow an exponential decline

$$dB_t = -\gamma B_t dt.$$

Secondly, the development stage is valued. At any point in time the $current$ per unit value of undeveloped reserve $X(V, T-t; D(Q))$ will be a function of the per unit value of a developed reserve V and per unit development cost $D(Q)$. By $current$ we here emphasize that the value of the developed reserve will fluctuate as the commodity price fluctuates. T denotes the expiration date and t denotes the current date; consequently $T-t$ refers to the remaining depletion time of the mine. A differential equation is introduced to find the value of the undeveloped reserve, $X(V, T-t; D(Q))$:

$$\frac{\partial X}{\partial t} = rX - (r-\delta)\frac{\partial X}{\partial V} - \frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 X}{\partial V^2}, \tag{2.3.1}$$

where r is the riskfree rate and δ is the payout rate of the developed reserve.

Finally, the exploration stage is valued. The exploration stage can be represented as the option to spend the expected exploration costs E^*, and receive the expected value of undeveloped reserves,

$$X^*(V) = \int QX(V, T-t; D(Q))dF(Q),$$

where Q represents the random quantity of recoverable hydrocarbons in the reservoir and $F(Q)$ denotes the probability distribution over the quantity of hydrocarbons. By using equation 2.3.1 and by assuming that development commences

immediately after the successful exploration has occurred, an expression for the current value of an option to receive a unit of developed reserves by paying the per unit combined expected exploration and development costs may be derived.

The final part of the paper presents some empirical results. The option valuation approach is used to estimate the market value of selected petroleum tracts (water expanses). The tracts were awarded to the oil industry from the Government in a federal lease sale and consist of 21 leases. The option valuation estimates are then compared with estimates using the Discounted Cash flow Method (DCF). The DCF estimates were prepared by the US Geology Service. Finally, both sets of estimates are compared with industry bids.

2.3.3 Implementing a real option model for valuing an undeveloped oil field

Cortazar and Schwartz (1997) implement a real option model for valuing an undeveloped oil field. Like Paddock et al. (1988) they model the development as a three stage process. Stage 1 represents the period before the firm commits to the development. Thus stage 1 can contain both exploration activities and postponement. Stage 2 and stage 3 represent development and production respectively. In order to value the oil field, stage 3 is valued first, then stage 2 and finally stage 1. This approach resembles a dynamic programming approach. Dynamic programming and stochastic dynamic programming are treated in section 3.2 and section 3.3. The value of an oil field in stage 3 is given by

$$\frac{1}{2}V_{SS}S^2\sigma^2 - q_pV_Q + M + (r - c + \beta(S - S^*))SV_S - (r + \lambda)V = 0, \tag{2.3.2}$$

subject to the boundary condition

$$V(S, 0) = 0. \tag{2.3.3}$$

The notation used in equations 2.3.2 and 2.3.2 is summarized in table 1. Appendix A contains details concerning the derivation of equation 2.3.2 and it also explains the reasoning behind the modelling and assumptions used in the article. Valuation expressions for the other stages are derived using the same arguments as those used in the derivation of equation 2.3.2.

Since the model presented does not have an analytical solution, a numerical solution is presented. Since the article presents a numerical solution to the problem, it could be argued that this contribution should be categorized as numerical. However, the article applies the framework of option pricing theory and erects from the analytical tradition. Therefore, it is natural to categorize it as an analytical contribution. The numerical method of finite differences is used. The results from a case study are presented in snapshots from an application written in Visual Basic and Fortran.

3 Stochastic programming approaches

3.1 Introduction

An oil field development project will include many kinds of real options in the different phases of the project. Management may for example be interested in postponing commencement of a project. Under certain circumstances it may be sensible to terminate or temporarily halt a project. Capacity adjustment may also represent an important real option to management. All these options are American; i.e. they may be exercised any time up to their expiration. To estimate the value of an American option the optimal exercise time - or the optimal stopping time - needs to be established. There is no closed form solution of an American option in the standard Black-Scholes framework. Numerical techniques have evolved as a response to this.

A fair share of the real option literature is found in the operational research literature. Stochastic programming is an important class of methods in operational research. We will here consider two techniques from the stochastic programming tradition, namely stochastic dynamic programming and scenario aggregation. In the next section we will consider various hybrid techniques which represents another important class of numerical approaches.

3.2 Dynamic stochastic programming

The statement

> *An optimal policy has the property that whatever initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

is known as *Bellman's Principle of Optimality*. This principle forms the basis for the fundamental equation of optimality which is central in dynamic programming. To understand this principle sufficiently, we first highlight some notions from dynamic programming. Important concepts in dynamic programming are $time\ horizon$, $state\ variables$, $decision\ variables$, $return\ functions$, $accumulated\ return\ functions$ and $transition\ function$. The $time\ horizon$ refers to the number of stages or time periods in the problem. The $state\ variables$ quantify the states of the system, for example oil production capacity. The $decision\ variables$ represent management's ability to make decisions, for example the decision to install new process equipment, to install injection wells or to shut down parts of a field or the entire field. A $return\ function$ reflects the immediate cost or benefit associated with the execution of a specific decision in a specific state. The $accumulated$

$return\ functions$ show the accumulated return from now and until the end of the time period associated with the execution of a specific decision in an specific state. The $transition\ function$ shows how the state variables change as a function of decisions. Remark 3.2.1 below deals with restrictions we may wish to impose on transition functions.

The application of Bellman's principle results in a decomposition of the optimization problem into an immediate return function and a continuation value:

$$V_n(i) = \max_a \{R(i,a) + \frac{1}{1+r} \sum_j p_{ij} \times V_{n+1}(j)\} \ \forall i, n = 1, \ldots, N-1 \qquad (3.2.1)$$

and

$$V_N(i) = \max_a \{R(i,a)\} \ \forall i, \qquad (3.2.2)$$

where the notation used is explained in table 2. The equations 3.2.1 and 3.2.2 are solved recursively starting at the final stage and working backwards in time one stage at a time towards the initial state. The advantage of the decomposition above is that only two values, i.e. the immediate return function and the continuation value, need to be compared at a given stage n. This is beneficial for computational purposes.

Remark 3.2.1 *In general we are looking for transition functions that possess certain desired properties. To understand what properties we seek we introduce the concept of a* $generator$*. If* $f(X_t)$ *is an appropriately smooth function on the state space and* X_t *is the state vector, then the operator* A *is a mapping that creates a new function* A_f *on the state space given by*

$$A_f(x) = \lim_{dt \to 0} E\{\frac{f(X_{t+dt}) - f(X_t)}{dt} | X_t = x\}. \qquad (3.2.3)$$

Heuristically the function $A_f(X_t)$ *is an "expected differentiation" of* $f(X_t)$ *given that* $X_t = x$*. The generator is then a mapping that maps* f *into a new function* A_f*, where both* f *and* A_f *are functions of* X_t*. Time is often discretized; then we set* $dt = 1$ *in equation 3.2.3 and we do not need the limit. We are looking for transition functions* f *that have the property*

$$\begin{aligned} A_f(x) = 0 &\iff E\{f(X_{t+1} - f(X_t) | X_t = x\} = 0 \\ &\iff E\{f(X_{t+1} | X_t = x\} = f(X_t) \end{aligned}$$

for all X_t *within the state space.*

To illustrate the mechanics of stochastic dynamic programming, consider the following example, inspired by Wallace and Kall (1994).

Variable	Representation
$V(S,Q)$	the value of the oil field at stage 3
S	oil spot price
Q	remaining reserves of oil
S^*	long term average unit price of oil
β	mean reversion parameter
r	risk free rate
c	mean convenience yield on holding one unit of oil
dW	increment in a standard Gaussian Wiener process
q_p	annual production rate stage 3
M	cash flowing to the owner of the field in stage 3 per unit of time
λ	risk premium associated with the country where the oil field is located

Table 1. Notation used in equation 2.3.2.

Variable	Representation
$V_n(i)$	the maximum expected value in state i at stage n
$R(i,a)$	the immediate return obtained by taking action a in state i
r	the rate of return per period,
$p_{ij}(a)$	the probability of going from state i to state j given action a
N	the final stage

Table 2. Notation used in equations 3.2.1 and 3.2.2.

Example 3.2.2 *As the manager of an oil platform in the Norwegian oil company Hydro you are considering to lease new processing equipment for your platform. In two years the platform will be shut down. If you keep the old equipment you will make a net profit of 7 % in the first year and 5.5 % in the second year. If you decide to install the new processing equipment you will make 8 % or 12 % in the first year with equal probability. The second year you will make 5 % or 9 % with equal probability. There will also be a lease cost associated with the new equipment amounting to 20 million NOK per year. To deinstall the new processing equipment there is a cost of 10 million NOK. As a manager you ask yourself the following question: Should you install the new equipment the first year, the second year or both years? We assume that if the new equipment is installed in any of the periods, it has to be de-installed at the end of year 2. Both alternatives will here increase the net wealth of the platform, but the risk and the reward of the two alternatives are different. Our preferences will therefore be closely related to our initial wealth. The greater initial wealth the more interested we are in taking on more risk. In this example we shall assume that our initial wealth S_0 is greater than 1000 million NOK. Figure 2 illustrates our situation. Now we have to define the model parameters. First we define*

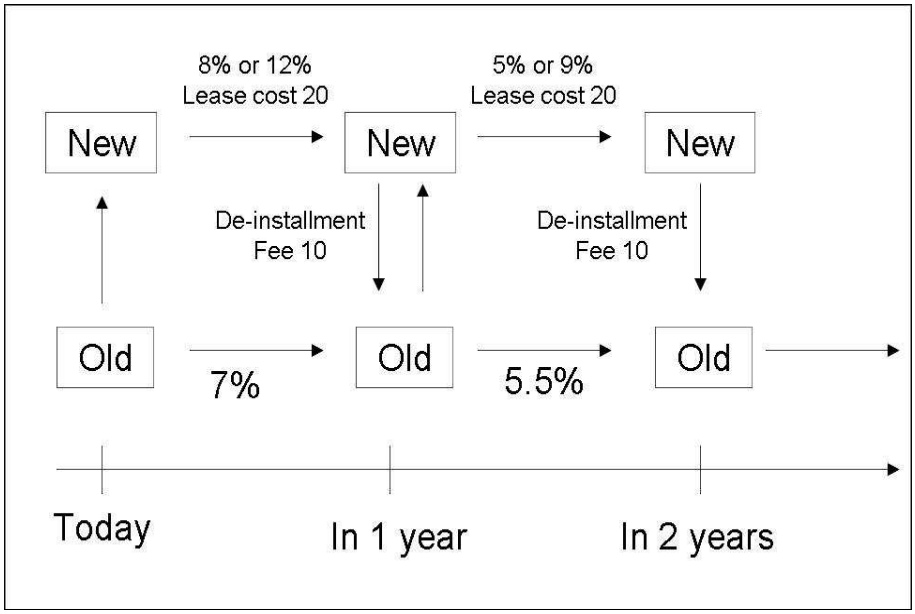

Figure 2. An illustration of example 3.2.2.

the two-dimensional state variables $z_t = (z_t^1, z_t^2)$. The first state variable z_t^1 refers to the type of processing equipment (New, Old); the second state variable z_t^2 refers to our wealth at time t. $z_t = (Old, S_t)$ represents a state where our wealth is S_t and we use the old processing equipment in stage t. Accumulated return functions will be denoted by $f_t(z_t^1, z_t^2, x_t)$. They describe how the amount z_t^2 will grow using processing equipment z_t^1, up to the end of the time horizon, if the processing equipment x_t is chosen in the next period, and optimal decisions are made thereafter. We are also interested in the optimal

accumulated return $f^*(z_t^1, z_t^2)$ for a given state, which is found by maximising all possible decisions.

We now have three stages that will be denoted Stage 0, Stage 1 and Stage 2. These stages represent today, 1 year from now and 2 years from now, respectively. Taking a closer look at the stages, we see that:

$Stage\ 2$: All we can do is to de-install the new processing equipment:

$$f_2^*(New, S_2) = S_2 - 10, \quad f_2^*(Old, S_2) = S_2.$$

$Stage\ 1$: We have to consider the two processing equipments separately. We look at the situation assuming we purchased the new processing equipment in stage 0. Now we can either stick with the new processing system for another year, making a net profit of $S_2 = 0.5 * \{S_1 * (1.05 - 20) + S_1 * (1.09 - 20)\}$, or we can de-install the new processing system and go back to the old processing system, making a profit of $S_2 = (S_1 - 10) * 1.055$. We need to calculate

$$f_1^*(New, S_1, New) = 1.07 * S_1 - 30\ , f_1^*(New, S_1, Old) = 1.055 * S_1 - 30,$$

yielding

$$f_1^*(New, S_1) = \begin{cases} 1.055 * S_1 - 10.5 & \text{if } S_1 < 1300 \\ 1.07 * S_1 - 30 & \text{if } S_1 \geq 1300. \end{cases}$$

Now assume that we did not purchase the new processing equipment in stage 0. Then we can either decide to install the new processing system or stick with the old the last year as well. This gives us

$$f_1^*(Old, S_1) = \begin{cases} 1.055 * S_1 & \text{if } S_1 < 2000 \\ 1.07 * S_1 - 30 & \text{if } S_1 \geq 2000. \end{cases}$$

$Stage\ 0$: Since we start out with the old processing system, we only need to check the situation from that perspective. We find that

$$f_0(Old, S_1, New) = \begin{cases} 1.1605 * S_0 - 31.6 & \text{if } S_0 < 1200 \\ 1.177 * S_0 - 51.4 & \text{if } S_0 \geq 1200 \end{cases} \tag{3.2.4}$$

and that

$$f_0(Old, S_1, Old) = \begin{cases} 1.12855 * S_0 & \text{if } S_0 < 1869.17 \\ 1.1449 * S_0 - 30 & \text{if } S_0 \geq 1869.17. \end{cases} \tag{3.2.5}$$

Comparing the expressions 3.2.4 and 3.2.5 we see that it is optimal to install the new processing system today and de-install it in two years.

3.3 Applications of dynamic programming

The dynamic programming approach requires that equation 3.2.1 is solved for all possible paths of every valid combination of state variables across the entire state space. As the number of state variables grows large, this becomes an unmanageable task for even the most resourceful computers. This phenomenon is referred to as "the curse of dimensionality". Although dynamic programming is widely used in operational research the number of contributions dealing with oil development projects is small. "The curse of dimensionality" is one important reason for this.

3.3.1 The PhD of Morten W. Lund

In his PhD thesis Lund (1997) aims to identify the value of flexibility in offshore oil development projects. Lund recognizes that traditional valuing techniques such as the discounted cash flow method have significant shortcomings that make them suboptimal for valuing oil field development projects. The weaknesses that are emphasized by Lund are listed and treated in section 1.

A prototype for an oilfield development project is developed. The life of the project is divided into phases. Some phases are aggregated and some simplifications are made. The main objective is to mirror the major decisions and information pattern throughout the project in the prototype. The aggregate project phases include exploration, conceptual study, engineering and construction and production. Different kinds of flexibility are discussed. In particular, the flexibility to postpone a project as well as the flexibility to terminate a project are discussed. Further, start/stop flexibility and capacity flexibility are treated. $Project\ finance$ refers to how the financing of a project is structured. Hedging strategies, the modelling of political risk and cash flow structuring are central issues in project finance. In Lund's thesis financial flexibility, i.e. issues related to project finance and the possible interaction between managerial flexibility and financial flexibility is not modelled.

Stochastic dynamic programming is used to evaluate the project. A state space of six variables of which three are stochastic is defined. The oil price, the reservoir volume and the well rate are modelled as stochastic variables. The oil price and the well rate are modelled using Markov processes; the reservoir volume is modelled with a 0-dimensional tank model. The prototype is then applied to a case study. The analysis reveals that flexibility might be of considerable value to the project. As flexibility is not additive, the importance of a joint analysis is highlighted. Capacity flexibility and initiation flexibility are identified as important aspects of the development.

3.4 Scenario aggregation

The principle of scenario aggregation is analogous to the technique of using event trees in decision making in the sense that each scenario represents a complete path in an event tree. Since each scenario is deterministic no value is given to flexibility. If weights were assigned to the different scenarios, the solution would inherit the same deficiency regarding flexibility evaluation. The principle of scenario aggregation aims to solve this problem through aggregation of solutions for different scenarios, see Wets (1989) or Wallace and Kall (1994). Wets (1989) shows that this overall solution will converge to the solution of the stochastic optimization problem. A central concept in scenario aggregation is the $progressive\ hedging\ algorithm$. Lund (1997) explains the role the progressive hedging algorithm plays in scenario aggregation:

> *Scenario aggregation commonly applies the progressive hedging algorithm to determine the optimal policy. The algorithm can roughly be described as an iterative two-step procedure. First the individual scenario problems are solved as separate (deterministic) problems. The solutions to these problems are then aggregated in order to obtain consistent solutions among the scenarios. If the deviation between the aggregate solution and the individual ones is above a predetermined limit, an updated penalty term is introduced in the individual optimization problems. The individual problems are solved again, and a new aggregated solution is obtained. These iterations continue until the deviation between the separate policies is acceptable.*

3.5 Applications of scenario aggregation

Applications of scenario aggregation is to a large extent found in the operational research literature. The most relevant application is summarized below. Other applications include Ovidiu and Dekker (2002), who apply the method in strategic airline fleet planning, and Dye (1994), who discusses scenario aggregation in the context of hydro-thermal power scheduling.

3.5.1 The PhD of Tore W. Jonsbraaten

Whereas Lund (1997) tries to replicate the entire life of an oil offshore development project, Jonsbraaten (1998) in his PhD thesis focuses on the production phase only. Where Lund is concerned with analyzing the contribution of flexibility in a project's value, Jonsbraaten emphasizes accurate modelling of the reservoir and its applications. The perspective of Lund is more economic than Jonsbraaten's more technical approach.

In the first part of the thesis various reservoir models are presented. The selected reservoir model is three dimensional and the flow through the reservoir is mod-

elled with partial differential equations. Finite differences are applied to estimate these difference equations by means of Taylor expansions. Decisions such as platform capacity, which wells to drill and production strategy for each well may be more thoroughly and adequately addressed in these more complex models. But a more complex model is not necessarily a better model, and a cost-benefit analysis and the purpose of the study should provide guidelines to model choice.

In the second part of the thesis four papers are presented. The first paper aims to optimize an oil field under price uncertainty. A finite set of oil price scenario and policy aggregation technique is applied. The reservoir is modeled by the use of a deterministic model. In the second article Jonsbraaten develops a class of stochastic programs with decision dependent random elements. The third paper deals with optimal selection and sequencing. Initially the reservoir is limited. When the first well is drilled, this information will be crucial in the following decision making. Dependent on the results from this drilling, further action is taken. Both the number of wells and the location of the wells may be optimized given this information. If it is decided to drill another well, the process is repeated. A Bayesian model for updating the a priori probability distribution over reservoir characteristics is proposed. Jonsbraaten also shows how this decision problem can be modelled by means of a decision three. The last article is concerned with oil extraction on a block boundary. A typical situation is two competing oil companies both eager to deplete the reservoir first. Jonsbraaten shows there exists a Nash equilibrium for this non-cooperative game; i.e. both oil companies can make simultaneously optimal decisions without co-operating.

4 Hybrid techniques

4.1 Introduction

One advantage of Monte Carlo simulation is that its convergence rate is independent of the number of state variables To be more precise, the approximation error in Monte Carlo simulation is of order $o(m^{-1/2})$ irrespective of the number of state variables. Here m denotes the number of observations in the simulated sample. In other methods, such as Riemann approximation, the approximation error with p state variables is of order $o(m^{-1/p})$. Another advantage is that it can easily handle a wide variety of models and payoff structures. Monte Carlo simulation was for a long time regarded to be not applicable to American-style pricing problems. The reason is that the state variables are simulated $forward$ in time. A path price is then estimated for a certain state trajectory and a $pre\text{-}specified$ exercise policy. An unbiased estimate of the security price may be obtained by averaging inde-

pendent samples of path prices. By contrast assets with early exercise features are generally priced by 'backward' algorithms. At maturity the optimal exercise strategy is easily determined, since all uncertainty is resolved at this point in time. The optimal exercise policy is then found by using dynamic programming proceeding backward in time. The difficulty in using simulation to price American options thus stemmed from applying a forward based algorithm to a problem that required a backward problem to solve.

Broadie and Glasserman (1997) propose a way to deal with this problem by developing two estimates of the asset price. One estimate is biased high and the other is biased low. Both estimates are asymptotically unbiased and converge asymptotically to the true asset price. Heuristically the two estimates together form a confidence interval for the asset price. This method will be explained in more detail in section 4.2 below. Glasserman (2004) proposes this technique and many others to estimate the value of American options. Some important aspects of many of these methods will be summarized in all the following subsections of section 4. Many of these methods uses the same basic approach in the sense that a high estimate and a low estimate are generated. Together these estimates form a confidence interval for the asset price. Longstaff and Schwartz (2001) approximate the conditional expectation in the dynamic programming principle by projections on a finite set of functions. Monte Carlo simulation and least squares regression is used to compute the value function of these projections. This technique is treated in Glasserman (2004) and is shown to be a special case of the stochastic mesh method[1]. Clement et al. (2002) prove almost sure convergence of the Longstaff and Schwartz algorithm under fairly general conditions.

4.2 Random Tree Methods

Two estimators are introduced, one biased high and one biased low; together the estimates form a confidence interval for the true asset price. The two estimators are created by constructing a random tree. The parameter b represents the number of branches per node. Figure 3 illustrates a random tree with $b = 3$. We will now explain how the high estimator Θ and the low estimator θ are constructed and state the intuition behind their high and low bias. [2].

4.2.1 The high estimator Θ

In the following we will assume that we have a finite amount of exercise opportunities; i.e. we are studying Bermudan options. We introduce C to represent

1. The stochastic mesh method is also treated in Glasserman (2004) and will be summarized in this section. Longstaff and Schwartz (2001) do, however, not construct a confidence interval in the spirit of Glasserman (2004) in their method.
2. The explanations are based on the explanations of Broadie and Glasserman (1997)

the price of an American call option with $d + 1$ exercise opportunities at time $t_i; i = 0, \ldots, d$ [3]. The definition of the high estimator Θ is the call value obtained by applying a dynamic programming algorithm to the random tree. At maturity, the option value is known, since all uncertainty is resolved at this point in time. At each prior date the option value is obtained by maximizing the immediate exercise value and the expectation of the succeeding discounted option values. The final option value of Θ is the estimated option value at the initial node. Intuitively, Θ is biased high due to perfect foresight; i.e. in the algorithm we know "everything" because we start at the end of the time period and work ourselves backwards. In real life we stand at a point in time and know the history up to this point and nothing about the future.

4.2.2 The low estimator θ

Branches at each node is divided into two sets, E_1 and E_2. The exercise decision is determined by means of the branches in the first set E_1. The continuation value is, if necessary, estimated by using the branches of the second set E_2. To understand why this estimator is biased low consider the time just prior to expiration. The exercise decision is now based on unbiased information from the maturity date. This exercise decision may now be right or wrong, due to the finiteness of the sample. To understand why the decision may be wrong, note that the finiteness of the sample will result in a loss of information. There will be points in the state space at this point in time that are not contained in the branches of the random tree. Therefore it is possible to infer a suboptimal decision based on the information in the random tree. If the correct decision is inferred from the information, the estimator would be unbiased. But if the exercise decision is wrong and a suboptimal decision is made, the value of the node will be an unbiased estimate of the lower value associated with the incorrect decision. The expected node value will then be biased low, since it is a weighted average of an unbiased estimate (based on the correct decision) and an estimate which is based low (based on the incorrect decision).

At each node, branch 1 is used to estimate the continuation value and the other $b - 1$ branches to estimate the exercise decision. This process is repeated $b - 1$ times. The b values obtained are averaged to determine the option value at the node.

3. With no dividends it is never optimal to exercise an American call option prior to maturity, see for example Hull (2000).

4.2.3 Properties of the estimators Θ and θ

In this section the estimators will be precisely specified. The following notation is used:

- $\{S_t : t = 0, 1, \ldots, T\}$ is a (possibly vector-valued) risk-neutralized Markov chain denoting all state variables.
- $\exp -R_t$ is the discount factor from $t-1$ to t.
- $h_t(s)$ is the payoff from exercise at time t in state s.
- $f_T(s) = h_T(s)$, i.e. at expiration the option is worth the payoff from immediate exercise.
- $g_t(s) = E\{\exp -R_t f_{t+1}(S_{t+1})|S_t = s\}$ is the continuation value at time t in state s.
- $f_t(s) = \max h_t(s), g_t(s)$ is the option value at time t in state s.

A random tree with b branches per node is represented by the array

$$\{S_t^{i_1 \ldots i_t} : t = 0, 1, \ldots, T \; i_j = 1, \ldots, b \; j = 1, \ldots, t\},$$

see figure 3 for an illustration.

4.2.4 The high estimator Θ

The high estimator Θ is defined recursively by

$$\Theta_T^{i_1 \ldots i_T} = f_T(S_T^{i_1 \ldots i_T})$$

and

$$\Theta_t^{i_1 \ldots i_t} = \max\left\{ h_t(S_t^{i_1 \ldots i_t}), \frac{1}{b} \sum_{j=1}^{b} \exp -R_{t+1}^{i_1 \ldots i_t j} \Theta_{t+1}^{i_1 \ldots i_t j} \right\},$$

for $t = 0, \ldots, T-1$. At each node this estimator chooses the maximum of the early exercise payoff and the continuation value estimated from all successor nodes. Let $\bar{\Theta}_0$ denote the sample mean of n independent replications of Θ_0. Then it is stated and shown in Broadie and Glasserman (1997) that $\bar{\Theta}_0(b)$ converges to $f_0(S_0)$ in probability and is thus a consistent estimator of the option value. Further, it is stated and shown in the same article that for finite b the bias is always positive; i.e. that $E\{\Theta_0(b)\} \geq f_0(S_0)$ for all b.

4.2.5 The low estimator θ

The low estimator θ is defined recursively as follows. First let

$$\theta_T^{i_1 \ldots i_T} = f_T(S_T^{i_1 \ldots i_T}).$$

Next define

$$\eta_t^{i_1 \ldots i_t j} = \begin{cases} h_t(S_t^{i_1 \ldots i_t} & \text{if } h_t(S_t^{i_1 \ldots i_t} \geq \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} e^{-R_{t+1}^{i_1 \ldots i_t i}} \theta_{t+1}^{i_1 \ldots i_t i} \\ e^{-R_{t+1}^{i_1 \ldots i_t i}} \theta_{t+1}^{i_1 \ldots i_t i} & \text{if } h_t(S_t^{i_1 \ldots i_t} < \frac{1}{b-1} \sum_{\substack{i=1 \\ i \neq j}}^{b} e^{-R_{t+1}^{i_1 \ldots i_t i}} \theta_{t+1}^{i_1 \ldots i_t i}, \end{cases}$$

for $j = 1, \ldots, b$. Then let

$$\theta_t^{i_1 \ldots i_t} = \frac{1}{b} \sum_{j=1}^{b} \eta_t^{i_1 \ldots i_t j},$$

for $t = 0, \ldots, T-1$. This estimator is also consistent. In Broadie and Glasserman (1997) it is shown that the low estimator has the same convergence properties as the high estimator; i.e. we have that $\bar{\theta}_0(b)$ converges to $f_0(S_0)$ in probability and is thus also a consistent estimator of the option value. Further, it is stated and shown in the same article that for finite b the bias is always negative; i.e. that $E\{\theta_0(b)\} \leq f_0(S_0)$ for all b.

4.2.6 Strengths and weaknesses of the random tree method

The random tree method is intuitive and easy to communicate to users. An important weakness of the method is that the computational work is exponential in the number of exercise opportunities.

4.3 Stochastic Mesh Methods

The stochastic mesh method is developed in Glasserman and Broadie (1997). Like the random tree method the stochastic mesh method estimates the value of an American option by solving a randomly sampled dynamic programming problem. Standing at node i the stochastic mesh method however utilizes all nodes at time $i+1$. By contrast, the random tree method utilizes only nodes that are direct successors of the current node. Thus the number of nodes at each point in time says fixed in the stochastic mesh method. In this way a $mesh$ is created instead of a tree. Figure 4 illustrates a stochastic mesh.

The first task in the method is to generate the stochastic mesh $X_t(i); i = 1, \ldots, b$ and $t = 1, \ldots, T$. Methods for construction of the mesh will be described below. The mesh estimator is defined as

$$\widehat{f}_T(X_T(i)) = h_T(X_T(i)),$$

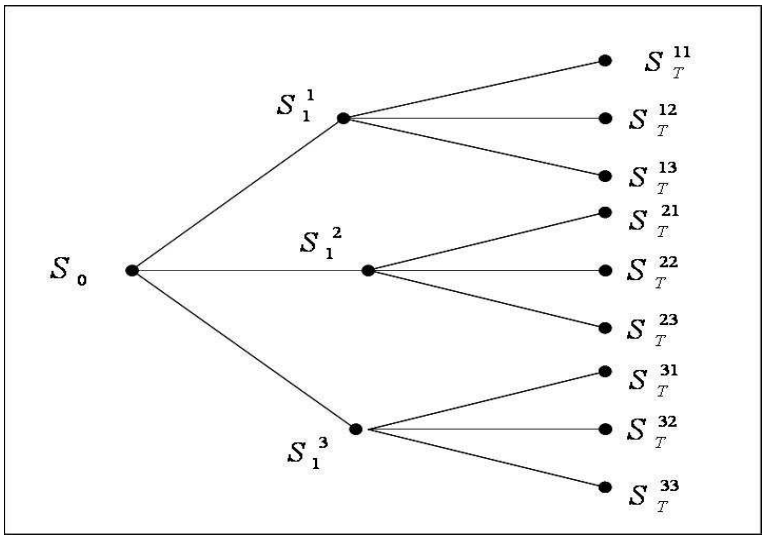

Figure 3. Random tree for $b = 3$

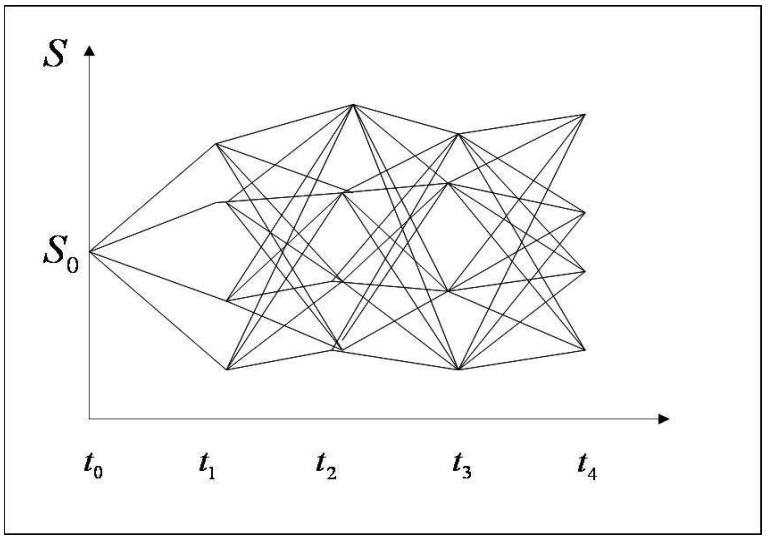

Figure 4. Mesh illustrated for $n = 1, T = 4$ and $b = 4$. A generic node in the mesh is denoted $X_t(i)$; a generic arc from one node to another has weight $w_t(X_t(i), X_{t+1}(j))$.

for $i = 1, \ldots, b$; $f_t(x)$ and $h_t(x)$ are the option value and payoff from exercise, respectively, at time t in state x. These functions were also defined in section 4.2.3 above. For time points $t = T - 1, \ldots, 0$ and $i = 1, \ldots, b$ the mesh estimator is

$$\widehat{f}_t(X_t(i)) = \max\{h_t(X_t(i)), \frac{1}{b}\sum_{j=1}^{b} \widehat{f}_{t+1}(X_{t+1})w_t(X_t(i), X_{t+1}(j))\}, \tag{4.3.1}$$

where $w_t(X_t(i), X_{t+1}(j))$ is a weight attached to the arc joining $X_t(i)$ to $X_{t+1}(j)$. The notation $\widehat{f}_t(X_t(i))$ represents the estimate of the true American value $f_t(X_t(i))$. Now we need to describe how the stochastic mesh is generated and how the weights on the arcs are determined.

Suppose that conditional on $S_t = x$, S_{t+1} has density $k_t(x, \cdot)$ and let $k_t(\cdot)$ denote the marginal density of S_t. The challenge then is to determine the density $l_t(x, \cdot)$ from which the vectors $X_t(i)$ in the mesh are sampled from. In Glasserman and Broadie (1997) it is shown that the weight

$$w_t(X_t(i), X_{t+1}(j)) = \frac{k_t(X_t(i), X_{t+1}(j))}{b^{-1}\sum_{l=1}^{b} k_t(X_t(l), X_{t+1}(j))} \tag{4.3.2}$$

is optimal, given that the mesh is Markovian[4]. Note that the numerator in equation 4.3.2 is simply the conditional density of S_{t+1} given that $S_t = x$. The denominator is heuristically the "average" probability of getting to node j at time $t+1$. The optimal weight of the mesh is thus the original conditional density of S_{t+1} given that $S_t = x$ scaled with a factor (the denominator). The purpose of the scaling is to avoid that the variance of the estimators grows exponentially with the number of exercise opportunities, see Glasserman and Broadie (1997). If the mesh is not Markovian, weights may be defined through likelihood ratios, see Glasserman and Broadie (1997) and Glasserman (2004).

4.3.1 The high estimator

The mesh estimator $\widehat{f}_t^b(X_t(i))$ defined in equation 4.3.1 represents the high estimator; b here denotes the size of the mesh. In Glasserman and Broadie (1997) it is stated and proved that the mesh estimator $\widehat{f}_t^b(X_t(i))$ is biased high and asymptotically unbiased. More formally, it is proved that $E\{\widehat{f}_0^b(S_0)\} \geq f_0(S_0)$ for all b and that $\widehat{f}_t^b(x)$ converges to $f_t(x)$ in probability for all x and t [5].

4.3.2 The low estimator

The low estimator is derived using a $path$ estimator. The path estimator is defined by simulating a trajectory of the underlying process S_t until the exercise region

4. Glasserman (2004) states the precise definition of a Markovian mesh.
5. Convergence in p norm is actually proved, but since this convergence is stronger, convergence in probability is implied.

determined by the mesh is reached. First the path $S = (S_0, S_1, \ldots, S_T)$ is simulated. This simulation is now independent of the mesh points. Along this path there will exist an optimal policy $\tau^*(S) = \min\{t : h_t(S_t) \geq f_t(S_t)\}$ for a payoff of the option $h_{\tau^*}(S_{\tau^*})$. The approximate optimal policy determined by the mesh exercises at

$$\widehat{\tau}(S) = \min\{t : h_t(S_t) \geq \widehat{f}_t(S_t)\}, \tag{4.3.3}$$

where $\widehat{f}_t(S_t)$ is given in equation 4.3.1. Define the path estimator, and thus the low estimator, as

$$\widehat{q} = h_{\hat{\tau}}(S_{\hat{\tau}}). \tag{4.3.4}$$

The stopping time defined in equation 4.3.3 is not necessarily optimal. To understand why, note that we only have a finite number of nodes, b, in the mesh at time i. Thus our mesh does not contain all available information at time i, or alternatively all points in the state space at time i. Thus the stopping time defined in equation 4.3.3 does not need to be an optimal stopping time since this stopping time is determined via the mesh. Consequentially $\widehat{q}$ in equation 4.3.4 represents a lower bound on the true asset price. In Glasserman and Broadie (1997) it is stated and proved that that the path estimator $\widehat{q}$ is biased low and asymptotically unbiased. More formally, it is proved that $E\{\widehat{q}^b\} \leq f_0(S_0)$ for all b and that $\widehat{q}^b$ converges to $f_0(S_0)$ in probability as $b \to \infty$.

4.3.3 Strengths and weaknesses of the stochastic mesh method

Since the number of nodes at each point in time says fixed, the computational work is not exponential in the number of exercise opportunities. This represents a clear advantage compared to the random tree method. In the stochastic mesh method it can however be challenging to find the weights of the mesh. In a Markovian mesh, the formula for the weights is given in equation 4.3.2. If the method of likelihood ratios is applied to find the weights, it is necessary to find a transition density. Transition densities for the underlying Markov chain may be unknown or may fail to exist. If, however, such a set of weights is found for the underlying Markov chain, it does not depend on the payoff functions h_i. Thus, the mesh can be applied to many different American options. See Glasserman (2004) for details regarding likelihood ratio weights.

4.4 Regression Methods

Longstaff and Schwartz (2001), and also Carriere (1996) and Tsitsiklis and Van Roy (1999) apply regression techniques to estimate continuation values from simulated paths. Continuation values are explained in section 3.2 and defined in section 4.2.3. The American options may then be priced by simulation. In particular, each option value $f_{t+1}(X_{t+1})$ defined in section 4.2.3 is regressed on the current state x. To be more specific, each continuation value $g_i(s)$ defined in section 4.2.3

is approximated by a linear combination of known functions of the current state. Then regression is applied to estimate the best coefficients of this approximation. A more precise expression for the continuation value is

$$g_i(x) = \sum_{r=1}^{M} \beta_{ir}\psi_r(x), \tag{4.4.1}$$

for some basis functions $\psi_r : \mathcal{R}^d \to \mathcal{R}$ and constants $\beta_{ir}, r = 1, \ldots, M$. Here d is the dimension of the state-space vector. Assuming that 4.4.1 holds, the vector β_i is given by

$$\beta_i = \{E\{\psi(X_i)\psi(X_i)^T\}\}^{-1} E\{\psi(X_i)f_{i+1}(X_{i+1})\} \equiv B_\psi^{-1} B_{\psi f}.$$

Here, B_ψ is an $M \times M$ matrix and $B_{\psi f}$ is an vector of length M. The coefficients β_{ir} may be estimated from observations of pairs $(X_{ij}, f_{i+1}(X_{i+1,j}))$, if we assume that $f_{i+1}(X_{i+1,j})$ is known. The least squares estimate of β_i is then given by $\widehat{\beta}_i = \widehat{B}_\psi^{-1}\widehat{B}_{\psi f}$, where $\widehat{B}_\psi^{-1}$ and $\widehat{B}_{\psi f}$ are the sample counterparts of B_ψ^{-1} and $B_{\psi f}$. A procedure for estimating the option value by regression is then[6]:

1. Simulate b independent paths $\{X_{1j}, \ldots, X_{mj}\}, j = 1, \ldots, b$ of the Markov chain.
2. At terminal nodes, set $f_{Tj} = f_T(X_{Tj}), j = 1, \ldots, b$.
3. Apply backward induction: For $i = m - 1, \ldots, 1$, if we have computed estimated values $\widehat{f}_{i+1,j}, j = 1, \ldots, b$, we then apply regression as described above to calculate $\widehat{\beta}_i = \widehat{B}_\psi^{-1}\widehat{B}_{\psi f}$ $\widehat{f}_{ij} = \max\{h_i(X_{ij}), \widehat{g}_i(X_{ij})\}, j = 1, \ldots, b$, with $\widehat{g}_i$ denoting the estimated continuation value of equation 4.4.1.
4. Set $\widehat{f}_0 = (\widehat{f}_{11} + \cdots + \widehat{f}_{1b})/b$.

As mentioned previously, the regression-based algorithm corresponds to a stochastic mesh estimator with an implicit choice of mesh weights. For an excellent comparison of these methods, interpretations, examples and treatment of implications, see Glasserman (2004).

4.4.1 The high estimator

The estimator $\widehat{f}_0 = (\widehat{f}_{11} + \cdots + \widehat{f}_{1b})/b$ represents the high estimator. It is not proved that $\widehat{f}_0$ is biased high. However, Clement et al. (2002) prove almost sure convergence of the Longstaff and Schwartz algorithm under fairly general conditions.

6. This procedure is stated in Glasserman (2004).

4.4.2 The low estimator

The estimated vectors of the coefficients β_i yield estimated continuation values $\widehat{g}_i(x)$ for every time point i and every state x. The reasoning we used in deriving the low estimator in the stochastic mesh in equation 4.3.4 is valid also for regression methods. The continuation values $\widehat{g}_i(x)$ define an exercise policy, which is suboptimal. Thus we have a low estimator.

4.4.3 Strengths and weaknesses of the regression method

The principle of using least-square regression to estimate the continuation value is intuitive and comprehendible. However, it may not be straight forward to choose the functions used in the regression. In the literature only some general recommendations are provided which may prove difficult to implement in a high-dimensional setting.

4.5 Parametric methods

We have now studied several techniques that try to approximate solutions to the dynamic programming problem presented in sections 4.2.3 and 3.2. This section will deal with a technique that tries to find the optimal stopping time within a parametric class. Consider the following example:

Example 4.5.1 *Brekke and Schieldrop (2000) consider a firm which has the option to build a plant that produces a single product. The plant uses one or two input factors with uncertain prices. They study a thermal power plant which is fuelled by either gas or oil. The plant can use either a pure technology that can use either gas or oil, or a flexible technology that can switch between the two types of fuel. Under the assumption that both fuel types are available, but only as pure technologies, the authors find explicit solutions to their optimization problem. When both fuel types are considered simultaneously, the firm is less willing to invest than in the benchmark case when the two options are considered separately. The findings of Brekke and Schieldrop are illustrated in figure 5.*

The firm will wait as long as the process lies in the continuation region C_{og}, but once the process hits the boundary ∂C_{og} the firm will invest and the choice of technology will be determined by which part of the boundary that is hit.

To be able to utilize the findings of the authors in a real world setting, it would is necessary to estimate the boundary of the continuation region ∂C_{og}. This task in itself may be formidable. If, however, we restrict ourselves to a parametric subclass, the task is much more tractable. Imagine for example that we limit ourselves to estimate boundary functions of the form

$$f(x) = \frac{1}{ax + b},$$

where x denotes the price of the fuel, i.e. oil or gas, and a and b are constants. Thus we try to find an optimal value within a $parametric\ class$.

Glasserman (2004) formalizes the parametric approach. A class of stopping rules, τ_θ is considered with each $\tau_\theta \in \mathcal{T}$ and Θ a subset of some $\mathcal{R}^M$. Here the set $\mathcal{T}$ represents all admissible stopping rules. Let

$$V_0^\theta = \sup_{\theta\in\Theta} E\{h_{\tau(\theta)}(X_{\tau(\theta))}\}.$$

We are now interested in estimating V_0^θ, the optimal value within the parametric class. Since the supremum in this definition is taken over a subset of all admissible stopping rules $\mathcal{T}$, it is true that

$$V_0^\theta \leq V_0 = \sup_{\tau\in\mathcal{T}} E\{h_\tau(X_\tau)\}. \tag{4.5.1}$$

If we can find a consistent estimator of V_0^θ, equation 4.5.1 tells us that it will underestimate the true option value. Glasserman (2004) gives an outline of an algorithm for the estimation of V_0^θ:

1. Simulate n_1 independent replications $X^{(j)}, j = 1, \ldots, n_1$ of the Markov chain $(X_0, X_1, \ldots, X_T)$.
2. Find θ by maximizing

   $$\widehat{V}_0^{\hat{\theta}} = \frac{1}{n_1}\sum_{j=1}^{n_1} h_{\tau^{(j)}(\hat{\theta})}(X^{(j)}_{\tau^{(j)}(\hat{\theta})}),$$

   where for each $\theta \in \Theta, \tau^{(j)}(\theta)$ is the time of exercise of the jth replication evaluated with parameter value θ.
3. Fix $\widehat{\theta}$ at the value found in step 2. Simulate n_2 additional independent replications of the Markov chain using the stopping rule $\tau_{\hat{\theta}}$ and compute the estimate

   $$\widehat{V}_0^\theta = \frac{1}{n_2}\sum_{j=n_1+1}^{n_1+n_2} h_{\tau^{(j)}(\hat{\theta})}(X^{(j)}_{\tau^{(j)}(\hat{\theta})}).$$

Step 3 is added to determine the direction of the bias of $\widehat{V}_0^{\hat{\theta}}$.

4.5.1 Strengths and weaknesses

Parametric methods reduce the optimal stopping problem to a tractable finite-dimensional optimization problem. Implementing the algorithm stated above can however be challenging. The main difficulty in implementing steps 1-2, beyond selection of the class of parametric rules, lies in the optimization in step 2. Glasserman (2004) gives some guidance on how this can be done.

4.6 State-space partitions

Whereas the dynamic program in the random tree method and the stochastic mesh method is based on randomly sampled states, the states in the state-space partition method are defined in advance. The definitions of the states are based on partitioning of the state space of the underlying Markov chain $X_0, X_1, \ldots, X_T$. The state space X_i is partitioned into b_i subsets $A_{i1}, \ldots, A_{ib_i}$ for $i = 1, \ldots, T$; i.e. for every exercise date. For the initial time 0, we set $b_0 = 1$ and $A_{01} = X_0$. Figure 6 illustrates a state-space partition.

Define the transition probabilities

$$p^i_{jk} = P(X_{i+1} \in A_{i+1,k} | X_i \in A_{ij})$$

for all $j = 1, \ldots, b_i, k = 1, \ldots, b_{i+1}$ and $i = 0, 1, \ldots, T$. For each $i = 1, \ldots, T$ and $j = 1, \ldots, b_i$ define

$$h_{ij} = E\{h_i(X_i) | X_i \in A_{ij}\}.$$

We now consider the backward induction

$$f_{ij} = \max\{h_{ij}, \sum_{k=1}^{b_{i+1}} p^i_{jk} f_{i+1,k}\}, \tag{4.6.1}$$

$i = 0, 1, \ldots, T-1, j = 1, \ldots, b_i$. Equation 4.6.1 is initialized with $f_{Tj} \equiv h_{Tj}$.

To implement this method we need to calculate the transition probabilities p^i_{jk} and average payoffs h_{ij}. To be able to do this, we simulate replications of the Markov chain $X_0, X_1, \ldots, X_T$ and estimate these quantities from the simulation. Glasserman (2004) provides a detailed description of the implementation. He produces an estimate that is guaranteed to have a low bias, relative to the true option value $f_0(X_0)$.

4.6.1 Strengths and weaknesses

The main difficulty to overcome using this approach lies in the selection of the state-space partition. If a partition could be applied to price many different American options, the effort could be justified. Intuitively, it is natural to expect that the estimated option value will converge to the true option value as the resolution of the partitions increases.

4.7 Applications of hybrid techniques

Since hybrid techniques are fairly new the number of relevant, interesting applications in the real option literature is scarce. Stentoft (2004) and Moreno and Navas (2003) have applied the least-squares regression method proposed by Longstaff and Schwartz to value some financial options of limited complexity.

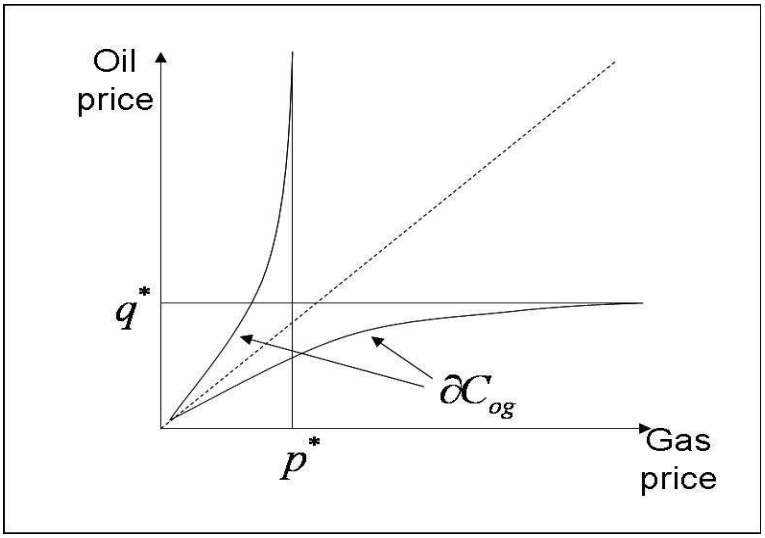

Figure 5. The border $\partial C_{og}$when both gas and oil technologies are available, but only as pure technologies. p^* and q^* are the reservation prices for the pure gas and the pure oil technologies. The $reservation\ price$ represents the price level at which the firm would invest, provided that only one of the technologies were available. If for example only oil were available, the firm would invest if the oil price were below or equal to q^* and else wait.

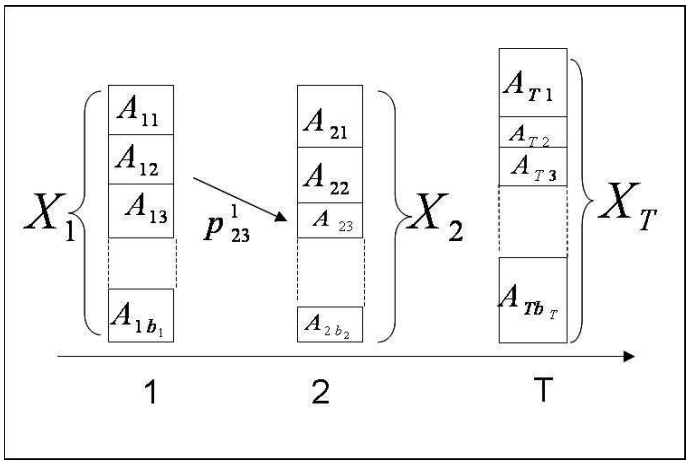

Figure 6. An illustration of a state-space partition.

4.7.1 The valuation of a copper mine

Cortazar et al. (2005) extend the model of Brennan and Schwartz (1985) summarized in section 2.3.1. Brennan and Schwartz applied a one-factor model to value a mine that could be open, closed or abandoned. Cortazar et al apply a three factor model to solve the same problem. Whereas Brennan and Schwartz (1985) only modelled the spot price of copper as a stochastic process, the model of Cortazar et al. (2005) has three state variables; the spot price S, the convenience yield y and the expected long term spot price return ν. Such a model is much more successful than a one-factor model in capturing the stochastic behaviour of commodity prices. Features of commodity prices are for example mean-reversion, declining volatility term structure and imperfect correlation between future returns with different maturities. This model extension became possible because Cortazar et al. (2005) apply Monte Carlo simulation to solve the valuation problem.

Cortazar et al. (2005) apply the least squares regression method proposed by Longstaff and Schwartz (2001). To reduce model complexity, a reduced-base implementation of the Longstaff and Schwartz method is proposed. To test the reduced-base form Cortazar et al. (2005) value an option that has a closed-form solution and compare the analytic solution to the alternative implementation of the LS01 method. Root Mean Square Error (RMSE) is then compared between the reduced-base form and other forms of polynomials. RMSE declines much more rapidly for the reduced-base form than other polynomial forms. Then the proposed model is implemented to solve the extended Brennan and Schwartz model. To validate the implemented model, the original Brennan and Schwartz model is solved with the proposed procedure. The results are very similar to the results obtained by Brennan and Schwartz. Daily values of the extended Brennan and Schwartz open mine is then plotted using historical copper prices as input in the valuation model. The mine value exhibits mean-reversion.

5 Concluding remarks

About 20 years ago the real options contributions were analytical in nature. As closed form solutions turned out to be difficult to produce for these analytical contributions, and as the development of the computer industry escalated, numerical methods became increasingly popular and in demand. In the past decade contributions applying hybrid techniques have emerged. The hybrid techniques try to overcome some deficiencies of the older numerical methods, like the 'curse of dimensionality' associated with stochastic dynamic programming.

Most contributions in the real option literature devote most of the energy on modelling commodity price uncertainty. Modelling of the reservoir uncertainty remains largely untouched. The potential for new, original contributions with more advanced and realistic reservoir models should therefore be considerable.

Acknowledgments

The knowledge and overview needed to write this summary were obtained during my Ph.D. studies at The University Of Oslo. I would like to express deep gratitude to my main supervisor Arne Bang Huseby, and to my co-supervisors Eivind Damsleth and Xeni Dimakos. This work was completed with the support of Norwegian Research Council.

A

A.1 Assumptions and models in application 2.3.3

The oil price is modelled as a Geometric Brownian Motion:

$$\frac{dS}{S} = (r - c + \beta(S^* - S))dt + \sigma dW. \tag{A.1}$$

The parameters in equation A.1 are explained in application 2.3.3. Some of the parameters in the model, like β, are most conveniently modelled by comparing historical prices for a futures contract for delivery of oil in τ years, $F(\tau, S)$, with those from a theoretical pricing equation. The value of an oil future is modelled by applying Ito's Lemma:

$$\begin{aligned} dF(S,\tau) &= \{F_\tau(-1) + \mu S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}\}dt + \sigma S dz F_s \\ &= \{-F_\tau + \frac{1}{2}\sigma^2 S^2 F_{ss}\}dt + \{\mu S dt + \sigma S dz\}F_s \\ &= F_s dS - F_\tau dt + \frac{1}{2}F_{ss}S^2\sigma^2 dt, \end{aligned}$$

where $\tau = T - t$ yields the minus in front of $F_\tau dt$. Further, $\mu = r - c + \beta(S^* - S)$, i.e. the expected capital gain in equation A.1. The oil market is assumed to be arbitrage free. Thus, an investor with a portfolio long in one unit of oil and short in $(F_s)^{-1}$ units of futures contracts has hedged his price risk and should earn the risk free rate:

$$dS + (c - \beta(S^* - S))dt - \frac{dF}{F_s} = rSdt. \tag{A.2}$$

Replacing A.1 in A.2 we get:

$$\begin{aligned} dS + (-\mu)Sdt - dS + \frac{F_\tau}{F_s}dt - \frac{1}{2F_s}F_{ss}S^2\sigma^2 dt = 0 \\ \iff \mu S F_s - F_\tau + \frac{1}{2}F_{ss}S^2\sigma^2 = 0 \\ \iff \frac{1}{2}F_{ss}S^2\sigma^2 + SF_s(r - c + \beta(S^* - S)) - F_\tau = 0. \end{aligned}$$

We are now interested in finding the value of the oil field in stage 3. By applying Ito's Lemma to the value of the oil field in stage 3, $V(S, Q)$, we obtain:

$$dV(S,Q) = V_s dS - q_p V_Q dt + \frac{1}{2}V_{ss}S^2\sigma^2 dt. \tag{A.3}$$

If an investor takes a long position in the oil field and V_s/F_s short position in futures contracts he hedges his risk and should earn the risk free rate plus a risk premium associated with the location of the oil field

$$dV + Mdt - \frac{V_s}{F_s}dF = (r + \lambda)V = 0, \tag{A.4}$$

where λ is explained in application 2.3.3. We now combine the expressions A.3 and A.4 to obtain:

$$\begin{aligned}
V_S dS - q_p V_Q dt + \frac{1}{2} V_{SS} S^2 \sigma^2 dt + M dt \\
-\frac{V_S}{F_S}\{F_S dS - F_\tau dt + \frac{1}{2} F_{SS} S^2 \sigma^2 dt\} - (r+\lambda) V dt = 0 \\
\iff V_S dS - q_p V_Q dt + \frac{1}{2} V_{SS} S^2 \sigma^2 dt + M dt \\
-V_S dS + V_S \frac{F_\tau}{F_S} dt - \frac{1}{2} F_{SS} S^2 \sigma^2 dt - (r+\lambda) V dt = 0 \\
\iff \frac{1}{2} V_{SS} S^2 \sigma^2 dt - q_p V_Q dt + M dt + V_S\{\frac{F_\tau}{F_S} dt - \frac{1}{2F_S} F_{SS} \sigma^2 S^2 dt\} - (r+\lambda) V dt = 0 \\
\iff \frac{1}{2} V_{SS} S^2 \sigma^2 - q_p V_Q + M + (r - c + \beta(S^* - S)) S V_S - (r+\lambda) V = 0,
\end{aligned} \tag{A.5}$$

which is equivalent to 2.3.2. We assume that dt is positive in equation A.5.

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