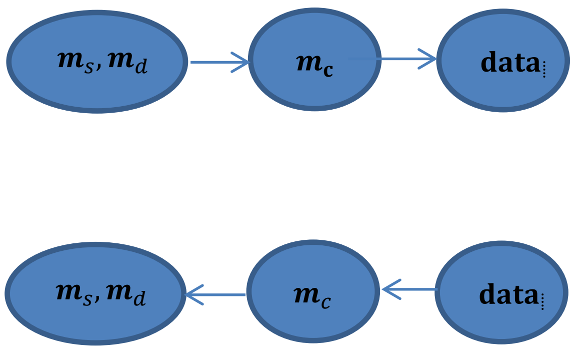
Note

# Using 3D inversion schemes to solve 4D inverse problems

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# Title — Using 3D inversion schemes to solve 4D inverse problems

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## Abstract

Seismic data observed at a given point in time are sensitive to the elastic parameters at this time. A Bayesian 3D inversion computes the posterior of the elastic parameters at that moment in time. In 4D inversion data at multiple time steps are collected, in order to investigate changes in elastic parameters. One option is to investigate each time step individually, and consider the differences, this approach ignore the temporal correlation of data. We use a Kalman filter approach to model the time correlations; in this approach we must separate the model for static and dynamic parameters. The current inversion methodology does not account for this separation. In the current note we show how we merge and split static and dynamic parameters, when conditioning to the current data set. This enables us to incorporate the standard inversion methodology for solving the 4D problem.


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# 1 Introduction

CRAVA is a program for seismic inversion which has been developed in cooperation between Statoil and NR since 2003. The program accounts for spatial continuity in earth parameters and assess the uncertainty in the inversion result. The initial idea of CRAVA was published in Buland, Kolbjørnsen and Omre 2003.  The seismic data entering the program is 3D volumes of pre-stack angle gathers origin from a single seismic survey. The purpose is this note is to generalize the method to work for 4D seismic, i.e. data from multiple seismic surveys collected over the same region in space, but with a time lag.  In the time between the seismic surveys the properties of the rock in the 3D volume has changed, either due to production of hydrocarbons or due to injection of CO2. In the note Kolbjørnsen and Kjønsberg 2011, the general framework for incorporating 3D inversion in a 4D scheme is developed. In this paper we discuss how to merge and split the information of static and dynamic parameters into and form the current elastic parameter.

# 2 Methodology

The back bone in CRAVA is a stochastic 3D inversion, which solves the linear inverse problem

$$\mathbf{d} \quad = \mathbf{G} \quad \mathbf{m} \quad + \boldsymbol{\varepsilon} \quad ,.$$

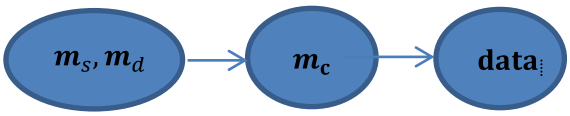
$$\mathbf{m} \quad \sim N(\boldsymbol{\mu} \quad , \boldsymbol{\Sigma} \quad )$$

$$\boldsymbol{\varepsilon} \quad \sim N(\mathbf{0}, \boldsymbol{\Gamma} \quad ),$$

under given stationarity conditions. In the 4D scheme presented in Kolbjørnsen and Kjønsberg (2011), this is the calculations that are made in the updating step, with one exception. In any step in the 4D scheme the elastic parameter $\mathbf{m}_\mathrm{c}$ in the equations above is a sum of two terms, one static and one dynamic,

$$\mathbf{m}_\mathrm{c} = \mathbf{m}_\mathrm{s} + \mathbf{m}_{\mathrm{d}.}$$

The current and dynamic parameters should be indexed with a time tag, but since we only consider one point in time, we drop this here. For the further progress of the 4D approach it is important to separate out the data impact on static and dynamic factors. It is off course not possible to  separate these components deterministic, but given the joint prior model of both static and dynamic variables, it is possible to compute the joint posterior.   We follow the used in the Kalman filer, see details in (Kolbjørnsen and Kjønsberg, 2011). The equation above has the dag:




We want to obtain the distribution of the static and dynamic variables given the data. We do this by computing the reverse arrows of the dag, and integrate out the elastic parameter. We present the general solution to this problem using matrix notation. This solution is off course not feasible to implement for high dimensional data as seismic amplitudes. However in CRAVA we do the computations in the Fourier domain where each frequency can be solved independently. To simplify notation we do not introduce the frequency-wavenumber index in the computations, but it is understood that all computations should be done for each frequency-wavenumber combination, thus the dimension of the current elastic parameters is 3.

The standard CRAVA computes the distribution

$$p(\mathbf{m}_{\mathrm{c}} \mid \mathrm{data}) = N\left(\mu_{\mathrm{c|data}}, \Sigma_{\mathrm{c|data}}\right)$$

This provides the inversion of the right arrow of the dag. Thus the next step is to provide the inverse of the left arrow. Let the joint distribution of $\mathbf{m}_{\mathrm{s}}$ and $\mathbf{m}_{\mathrm{d}}$ be given as:

$$\begin{bmatrix} \mathbf{m}_{\mathrm{s}} \\ \mathbf{m}_{\mathrm{d}} \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_{\mathrm{s}} \\ \mu_{\mathrm{d}} \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathrm{ss}} & \Sigma_{\mathrm{sd}} \\ \Sigma_{\mathrm{ds}} & \Sigma_{\mathrm{dd}} \end{bmatrix} \right)$$

This distribution is defined through the prior model formulation discussed in Kjønsberg and Kolbjørnsen (2011), and the Kalman filter relations in Kolbjørnsen and Kjønsberg (2011). We now investigate the joint distribution of $\boldsymbol{m}_{\mathrm{c}}, \boldsymbol{m}_{\mathrm{s}}$, and $\boldsymbol{m}_{\mathrm{d}}$:

$$\begin{bmatrix} \mathbf{m}_{\mathrm{c}} \\ \mathbf{m}_{\mathrm{s}} \\ \mathbf{m}_{\mathrm{d}} \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_{\mathrm{c}} \\ \mu_{\mathrm{s}} \\ \mu_{\mathrm{d}} \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathrm{c}} & \Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}} & \Sigma_{\mathrm{sd}} + \Sigma_{\mathrm{dd}} \\ \Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}} & \Sigma_{\mathrm{ss}} & \Sigma_{\mathrm{sd}} \\ \Sigma_{\mathrm{ds}} + \Sigma_{\mathrm{dd}} & \Sigma_{\mathrm{ds}} & \Sigma_{\mathrm{dd}} \end{bmatrix} \right)$$

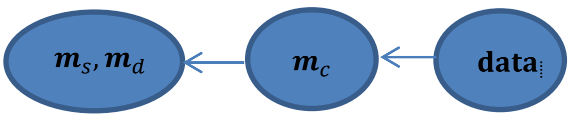
With:

$$\mu_{\mathrm{c}} = \mu_{\mathrm{s}} + \mu_{\mathrm{d}}$$
$$\Sigma_{\mathrm{c}} = \Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}} + \Sigma_{\mathrm{sd}} + \Sigma_{\mathrm{dd}}$$

The covariance matrix in the joint distribution is singular, but this will not influence the computations below. The conditional distribution of static and dynamic component given the distribution of the sum is given by standard relations of Gaussian inversion.

$$\begin{bmatrix} \mathbf{m}_{\mathrm{s}} \\ \mathbf{m}_{\mathrm{d}} \end{bmatrix} \Big| \mathbf{m}_{\mathrm{c}} \sim N\left( \begin{bmatrix} \mu_{\mathrm{s}|m} \\ \mu_{\mathrm{d}|m} \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathrm{ss}|m} & \Sigma_{\mathrm{sd}|m} \\ \Sigma_{\mathrm{ds}|m} & \Sigma_{\mathrm{dd}|m} \end{bmatrix} \right)$$
$$\mu_{\mathrm{s}|m} = \mu_{\mathrm{s}} + \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\mathbf{m}_{\mathrm{c}} - \mu_{\mathrm{c}}\right)$$
$$\mu_{\mathrm{d}|m} = \mu_{\mathrm{d}} + \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\mathbf{m}_{\mathrm{c}} - \mu_{\mathrm{c}}\right)$$
$$\Sigma_{\mathrm{ss}|m} = \Sigma_{\mathrm{ss}} - \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}}\right)$$
$$\Sigma_{\mathrm{dd}|m} = \Sigma_{\mathrm{dd}} - \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{sd}}\right)$$
$$\Sigma_{\mathrm{sd}|m} = \Sigma_{\mathrm{sd}} - \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{sd}}\right)$$
$$\Sigma_{\mathrm{ds}|m} = \Sigma_{\mathrm{ds}} - \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}}\right)$$

This distribution is again singular, to acknowledge that the distribution is singular, compute the distribution of $\boldsymbol{m}_{\mathrm{s}} + \boldsymbol{m}_{\mathrm{d}}$, and see that the mean of this is $\boldsymbol{m}$ and the variance is $\mathbf{0}$. This provides the inversion of the left arrow in the dag, and thus allows us to use the following dag:



The sought distribution is that of $\boldsymbol{m}_{\mathrm{s}}$ and $\boldsymbol{m}_{\mathrm{d}}$ given the data. This can now be computed by integrating out $\boldsymbol{m}_c$ using the inverse relations above. Thus the inverse relation is:

$$
\begin{aligned}
&\begin{bmatrix}\mathbf{m}_{\mathrm{s}}\\ \mathbf{m}_{\mathrm{d}}\end{bmatrix} \Big| \ \text{data} \sim N\left(\begin{bmatrix}\mu_{\mathrm{s|data}}\\ \mu_{\mathrm{d|data}}\end{bmatrix}, \begin{bmatrix}\Sigma_{\mathrm{ss|data}} & \Sigma_{\mathrm{sd|data}}\\ \Sigma_{\mathrm{ds|data}} & \Sigma_{\mathrm{ss|data}}\end{bmatrix}\right)\\
&\mu_{\mathrm{s|data}} = \mu_{\mathrm{s}} + \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\mu_{\mathrm{c|data}} - \mu_{\mathrm{c}}\right)\\
&\mu_{\mathrm{d|data}} = \mu_{\mathrm{d}} + \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\Sigma_{\mathrm{c}}^{-1}\left(\mu_{\mathrm{c|data}} - \mu_{\mathrm{c}}\right)\\
&\Sigma_{\mathrm{ss|data}} = \Sigma_{\mathrm{ss}} - \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\left(\Sigma_{\mathrm{c}}^{-1} - \Sigma_{\mathrm{c}}^{-1}\Sigma_{\mathrm{c|data}}\Sigma_{\mathrm{c}}^{-1}\right)\left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}}\right)\\
&\Sigma_{\mathrm{dd|data}} = \Sigma_{\mathrm{dd}} - \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\left(\Sigma_{\mathrm{c}}^{-1} - \Sigma_{\mathrm{c}}^{-1}\Sigma_{\mathrm{c|data}}\Sigma_{\mathrm{c}}^{-1}\right)\left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{sd}}\right)\\
&\Sigma_{\mathrm{sd|data}} = \Sigma_{\mathrm{sd}} - \left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{sd}}\right)\left(\Sigma_{\mathrm{c}}^{-1} - \Sigma_{\mathrm{c}}^{-1}\Sigma_{\mathrm{c|data}}\Sigma_{\mathrm{c}}^{-1}\right)\left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{sd}}\right)\\
&\Sigma_{\mathrm{ds|data}} = \Sigma_{\mathrm{ds}} - \left(\Sigma_{\mathrm{dd}} + \Sigma_{\mathrm{ds}}\right)\left(\Sigma_{\mathrm{c}}^{-1} - \Sigma_{\mathrm{c}}^{-1}\Sigma_{\mathrm{c|data}}\Sigma_{\mathrm{c}}^{-1}\right)\left(\Sigma_{\mathrm{ss}} + \Sigma_{\mathrm{ds}}\right)
\end{aligned}
$$

Again notice that if one consider the posterior distribution of the sum of $\boldsymbol{m}_{\mathrm{s}}$ and $\boldsymbol{m}_{\mathrm{d}}$, we get back the posterior distribution of $\boldsymbol{m}_c$ The center part of the reduction of the covariance matrix has the form:

$$\left(\Sigma_{\mathrm{c}}^{-1} - \Sigma_{\mathrm{c}}^{-1}\Sigma_{\mathrm{c|data}}\Sigma_{\mathrm{c}}^{-1}\right) = \Sigma_{\mathrm{c}}^{-1}\left(\Sigma_{\mathrm{c}} - \Sigma_{\mathrm{c|data}}\right)\Sigma_{\mathrm{c}}^{-1}$$

We see that if the posterior of the elastic parameter equals the prior, then there is no reduction in the posterior for neither the static nor the dynamic part.

# 3 Small examples

We look at two small examples, using the dimension 3 for all involved variables. This is the size of the variables in the Fourier domain. The functions split and merge are implemented in Matlab and supplied in appendix. We consider the two prior distributions, with mean and covariance:

$$
\begin{pmatrix}1.0\\2.0\\3.0\\3.0\\2.0\\1.0\end{pmatrix},
\begin{pmatrix}
3.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0\\
0.0 & 2.0 & 0.0 & 0.0 & 1.0 & 0.0\\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.5\\
0.5 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0\\
0.0 & 1.0 & 0.0 & 0.0 & 2.0 & 0.0\\
0.0 & 0.0 & 1.5 & 0.0 & 0.0 & 3.0
\end{pmatrix}
$$

and

$$\begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 3.0 \\ 2.0 \\ 1.0 \end{pmatrix}, \begin{pmatrix} 3.0 & 0.5 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.5 & 2.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 1.5 \\ 0.5 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 0.0 & 0.0 & 3.0 \end{pmatrix}$$

Note the only difference is the covariance in the cell (1,2) , and (2,1). The current prior distribution is then respectively:

$$\begin{pmatrix} 4.0 \\ 4.0 \\ 4.0 \end{pmatrix}, \begin{pmatrix} 5.0 & 0.0 & 0.0 \\ 0.0 & 6.0 & 0.0 \\ 0.0 & 0.0 & 7.0 \end{pmatrix}$$

$$\begin{pmatrix} 4.0 \\ 4.0 \\ 4.0 \end{pmatrix}, \begin{pmatrix} 5.0 & 0.5 & 0.0 \\ 0.5 & 6.0 & 0.0 \\ 0.0 & 0.0 & 7.0 \end{pmatrix}$$

Assume that the posterior distribution for the current elastic parameter is:

$$\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 3.0 \end{pmatrix}$$

for both priors. We find the posterior distributions to be

$$\begin{pmatrix} -1.800 \\ 0.000 \\ 1.571 \\ 1.800 \\ 0.000 \\ -1.571 \end{pmatrix}, \begin{pmatrix} 1.040 & 0.000 & 0.000 & -0.340 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.490 & 0.000 & 0.000 & 0.582 \\ -0.340 & 0.000 & 0.000 & 0.640 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.582 & 0.000 & 0.000 & 1.347 \end{pmatrix}$$

and

$$\begin{pmatrix} -1.891 \\ -0.185 \\ 1.571 \\ 1.891 \\ 0.185 \\ -1.571 \end{pmatrix}, \begin{pmatrix} 1.034 & 0.136 & 0.000 & -0.336 & -0.085 & 0.000 \\ 0.136 & 0.982 & 0.000 & -0.085 & 0.010 & 0.000 \\ 0.000 & 0.000 & 0.490 & 0.000 & 0.000 & 0.582 \\ -0.336 & -0.085 & 0.000 & 0.639 & 0.035 & 0.000 \\ -0.085 & 0.010 & 0.000 & 0.035 & 0.998 & 0.000 \\ 0.000 & 0.000 & 0.582 & 0.000 & 0.000 & 1.347 \end{pmatrix}$$

respectively. Comparing the results we see that when both the prior and posterior covariance of the current seismic parameter is independent, the problem is isolated for each elastic parameter. When we introduce correlation between two of them, we find that this influence both the posterior mean and covariance of these two parameters, whereas the third parameter remains unchanged. In general the correlation between static and dynamic parameters is lower in the posterior than in the prior. This is natural since an exact observation of the current variable, would give perfect negative correlation between static and dynamic parameters.

# 4 Conclusion

The decomposition presented in this note makes integration of 4D inversion in CRAVA simple. We only need to provide the prior mean and covariance as input to CRAVA. CRAVA can then invert in a standard fashion. After the CRAVA inversion we can use the prior and posterior mean and covariance to determine the posterior mean, variance, and covariance of the static and dynamic parts.

# References

Buland, Kolbjørnsen and More (2003), «Rapid Spatially Coupled AVO Inversion in the Fourier Domain». *Geophysics* ;Volum 68.(1) s. 824-836

Kjønsberg and Kolbjørnsen (2011), «Making a 4D seismic prior model from rock physics relations», Norsk Regnesentral 2011 15 s. NR Notat(SAND/22/2011)

Kolbjørnsen, Kjønsberg (2011) NR note to come. "Joint 4D inversion of multiple data sources for CO2 monitoring" Norsk Regnesentral 2011. NR Notat(SAND/20/2011)

# Appendix: Matlab code

```
function [mu_c Sigma_c]=mergeMeanAndVariance(mu_b,Sigma_b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% input:
%   mu_b:      Prior expectation of stationary and dynamic elastic parameters
%   Sigma_b:   Prior covariance of stationary and dynamic elastic parameters
%
%  output:
%   mu_c:     Prior expectation of current elastic parameters
%   Sigma_c:  Prior covariance of current elastic parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n = size(Sigma_b,1);

if(n/2~=round(n/2))
  Error= 'Input of wrong dimension (not even)' %#ok<NASGU>
else
  m=n/2;
  A=[eye(m) eye(m)];
  mu_c=A*mu_b;
  Sigma_c =  A*Sigma_b*A';
end;



function [mu_bd Sigma_bd]= splitMeanAndVariance(mu_b, Sigma_b,mu_cd, Sigma_cd)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  input:
%   mu_b:      Prior expectation of stationary and dynamic elastic parameters
%   Sigma_b:   Prior covariance of stationary and dynamic elastic parameters
%   mu_cd:     Posterior expectation of current elastic parameters
%   Sigma_cd:  Posterior covariance of current elastic parameters
%
%  output:
%    mu_bd:     Posterior expectation of stationary and dynamic elastic parameters
%    Sigma_bd:  Posterior covariance of stationary and dynamic elastic parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n = size(Sigma_b,1);
m = size(Sigma_cd,1);

if(n~=2*m)
  Error= 'Missmatch of dimensions' %#ok<NOPRT,NASGU>
else
  [mu_c  Sigma_c]=mergeMeanAndVariance(mu_b,Sigma_b);
   if(min(eig(Sigma_c-Sigma_cd)>=0) )

    A=[eye(m) eye(m)];
    Sigma_bc=Sigma_b*A';

    mu_bd = mu_b+Sigma_bc*(Sigma_c\(mu_cd-mu_c)) ;
    h1=(Sigma_bc/Sigma_c);
    Sigma_bd= Sigma_b - h1*(Sigma_c-Sigma_cd)*h1' ;

   else
     Error = 'Prior and posterior variance are inconsistent' %#ok<NOPRT,NASGU>
   end;
end;
```