FINITE ELEMENT DISCRETIZATIONS OF ELLIPTIC PROBLEMS IN THE PRESENCE OF ARBITRARILY SMALL ELLIPTICITY; AN ERROR ANALYSIS

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Abstract. The purpose of this paper is to analyse the error of the finite element method applied to the pressure equation arising in reservoir simulation. We study self-adjoint second order elliptic equations with discontinuous coefficients and of arbitrarily small (but uniformly positive) ellipticity. Under proper conditions on the permeability functions and the source term, we prove error estimates that are independent of the lower bound δ of the materiel coefficients. These results are based on an extensive regularity analysis of the interface problems of concern. More precisely, we show that the solution of our model problem is piecewise smooth, and that the associated Sobolev norms are bounded independently of δ. Finally, the error analysis is illustrated by numerical experiments.

Key words. reservoir simulation, second order elliptic equations, the finite element method, error analysis, discontinuous coefficients.

AMS subject classifications. 35J25, 65N12, 65N15.

1. Introduction. Consider the following prototypical elliptic boundary value problem

$$\begin{aligned} \nabla \cdot (K \nabla u) &= f \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where K is a given uniformly positive and bounded function defined on Ω. Let u_h denote an approximation of u computed by the finite element method. Given proper conditions on the finite element space V_h, it is well-known that

$$\|u - u_h\|_{H^1(\Omega)} \leq \sqrt{1+c} \sqrt{\frac{\sup_{x\in\Omega} K(x)}{\inf_{x\in\Omega} K(x)}} \inf_{q_h \in V_h} \|u - q_h\|_{H^1(\Omega)}, \tag{1.2}$$

where c is a constant only depending on the solution domain Ω (the constant appearing in Poincaré's inequality), see for instance [4]. Hence, u_h will be a good approximation of u provided that K has small variation and that the finite element space V_h is sufficiently large.

In this paper we will consider elliptic problems of the form (1.1) arising in reservoir simulation. For these type of models, K typically has large jump discontinuities and varies from[1] $10^{-6} - 10^2$. Hence, in such cases inequality (1.2) indicates that some sort of problem may arise for the efficient, and accurate, numerical solution of (1.1). Consequently, it might be necessary to apply adaptive methods, cf. e.g. [14] and references therein, in order to obtain acceptable results. Typically, mesh refinements are needed close to the discontinuities of K and in regions where K is close to zero.

However, under proper conditions on K and on the source term f we will prove an error bound of the form

$$\|u - u_h\|_{H^1(\Omega)} \leq c \inf_{q_h \in V_h} \|u - q_h\|_{H^1(\Omega)},$$

where c is a constant not depending on the lower bound δ of K. More precisely, such results are obtainable if the source term f vanishes in the low-permeable zones

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[1] We will always assume that the problem has been transformed to be dimensionless. Units for the physical quantities are therefore omitted.

of the reservoir, i.e. in the regions where K is close to zero. We will also assume that the domain Ω can be partitioned into subdomains such that the variation of K is relatively small in each subdomain. At the boundaries of these subdomains we assume that K has jump discontinuities. Moreover, given proper smoothness assumptions on K and the solution domain Ω, we prove that the solution of a problem of this form is piecewise smooth and that the following error estimate holds

$$\|u - u_h\|_{H^1(\Omega)} \leq ch.$$

Here, h represents the mesh parameter associated with the finite element space V_h, and c is again a constant independent of the lower bound δ of K. Thus, in such cases it seems like no grid refinements, due to small ellipticity and jumps in the coefficients, are needed.

Elliptic boundary value problems of the form (1.1) arise in a series of applications. Their mathematical properties have been thoroughly studied by several authors: Dautray and Lions [10], Gilbarg and Trudinger [16], Hackbusch [19] and Marti [23], to name a few. References and reviews of numerical methods for such problems, including error analyses, can be found in e.g. Bramble [1], Chan and Mathew [7], Ciarlet [8] and Hackbusch [18]. We would also like to refer to Dryja [12] and Dryja, Sarkis and Widlund [13] for their analysis of multilevel methods for elliptic problems with discontinuous coefficients.

The remainder of this paper is organized as follows: In the next section we describe our model problem and the necessary assumptions on the physical parameters. Section 3 contains the notation used throughout this paper and the discretization of our model problem. Sections 4-6 contain the theoretical results, and in Section 7 we present our numerical experiments.

2. The model problem. Let P represent the unknown fluid pressure related to steady state or incompressible flow in a heterogeneous reservoir, g the gravitational constant, ρ the density of the fluid and D the depth of the reservoir measured in the direction of gravity. Then the pressure equation arising in reservoir simulation can be written in the form

$$\nabla \cdot [\Lambda (\nabla P - \rho g \nabla D)] + \frac{q}{\rho} = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.1}$$

see for instance Ewing [15] or Peaceman [25]. In (2.1) Λ is the mobility tensor representing the viscosity of the fluid and the permeability of the reservoir. Source terms, such as injection and production wells located inside Ω, are incorporated in the model (2.1) by the function q.

Throughout the paper we will assume that the domain Ω is a union of two disjoint subdomains Ω_1, Ω_δ and a common boundary $\partial\Omega_\delta$. Here, $0 < \delta \ll 1$ is a small constant and Ω_δ represents a low-permeable zone in the reservoir. That is, $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$ and we assume that the mobility tensor Λ_δ has the form

$$\Lambda_\delta(x) = \begin{cases} \Lambda(x) & \text{for } x \in \Omega_1 \\ \delta\Lambda(x) & \text{for } x \in \Omega_\delta, \end{cases} \tag{2.2}$$

where Λ is a $O(1)$ mobility tensor defined on Ω. Clearly, by (2.2) Λ_δ is a mobility tensor of order $O(1)$ and $O(\delta)$ in Ω_1 and Ω_δ, respectively. A solution domain of this type is shown in Figure 2.1.

In this paper we will assume that g and ρ are constant over the domain Ω. Then, by putting $f = q/\rho$ and $p = P - \rho g D$ we can rephrase our model problem (2.1) in the following form

$$\nabla \cdot (\Lambda_\delta \nabla p) + f = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.3}$$

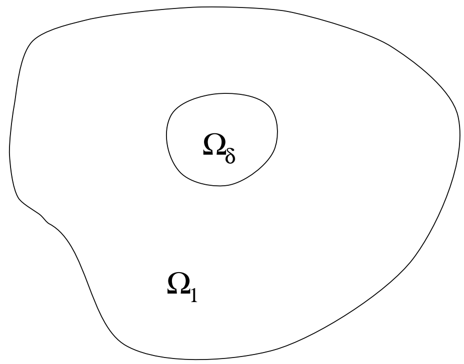

FIG. 2.1. *An example of a solution domain Ω consisting of two subdomains Ω_1,Ω_δ and a common boundary $\partial\Omega_\delta$. We assume that the mobility tensor Λ_δ is of order $O(1)$ and $O(\delta)$ in Ω_1 and Ω_δ, respectively.*

with a homogeneous Dirichlet boundary condition

$$p = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

Next, consider the source term $f = q/\rho$ and recall that q represents either injection or production wells located inside Ω. Injecting water into low permeable regions, i.e. zones containing hard rocks, requires a very high pressure. Hence, it is not desirable to position injection wells at such locations. Furthermore, fluids tend to flow around low permeable zones, and thus production wells should not be drilled in these regions. Therefore, from a physical point of view, we find it reasonable to assume that $q = 0$ in the area of low permeability. That is, we will assume that

$$f|_{\Omega_\delta} = 0 \tag{2.5}$$

throughout this paper.

Now, the purpose of this paper can roughly be formulated as follows; Let p_h be an approximation of the weak solution p of (2.2)-(2.5) computed by the finite element method. Then we want to prove error estimates for $p-p_h$, measured in proper Sobolev norms, that are independent of the lower bound δ of the mobility Λ_δ. More precisely, there exists a constant c, independent of δ and the mesh parameter h, such that

$$\|p - p_h\|_{H^1(\Omega)} \leq c \inf_{q_h \in V_h} \|p - q_h\|_{H^1(\Omega)}, \tag{2.6}$$

and

$$\|p - p_h\|_{H^1(\Omega)} \leq ch. \tag{2.7}$$

As we will see below, such estimates are obtainable because f is assumed to satisfy (2.5).

Normally, problems of the form (2.2)-(2.4) involving discontinuous materiel coefficients are referred to as interface problems. Interface problems of this kind have been analysed by several authors, cf. [11, 19, 20, 22, 26, 27]. However, to our knowledge error estimates independent of the lower bound δ of the mobility Λ_δ have not been established earlier.

Remarks.

1. For the sake of simplicity we will only consider homogeneous Dirichlet boundary conditions, see (2.4). However, it should be noted that our results are also valid if more general boundary conditions are applied.

2. The analysis presented in this paper can be extended to the case of a finite number of subdomains Ω_{δ_i} with order $O(\delta_i)$ mobility. In this case, condition (2.5) must be replaced by the assumption that f is equal to zero in each of these subdomains.
3. It is straight forward to prove similar results in the case of three space dimensions.
4. In [24] we analysed the convergence properties of p as $\delta \to 0$ for problems of the form (2.2)-(2.5), cf. also [5]. Moreover, in [6] we studied a preconditioner for the efficient numerical solution of problems of this kind.

3. Weak formulation and discretization. To get a well-posed variational problem of (2.3)-(2.4) we assume that $f \in L^2(\Omega)$ and that the mobility tensor $\Lambda(x) = (\lambda_{i,j}(x))$ is a symmetric uniformly positive definite matrix satisfying

$$\lambda_{i,j} \in L^\infty(\Omega) \quad \text{for } i,j = 1,2, \tag{3.1}$$

$$0 < m \leq \frac{\mathbf{z}^T \Lambda(x) \mathbf{z}}{|\mathbf{z}|^2} \leq M \quad \text{for all } \mathbf{z} \in \mathbb{R}^2 \setminus \{0\} \text{ and } x \in \Omega. \tag{3.2}$$

Here, m and M are finite constants independent of δ, and $|\mathbf{z}|$ denotes the Euclidean norm of $\mathbf{z} \in \mathbb{R}^2$.

Next, $H^1(\Omega)$ denotes the classical Sobolev space of square-integrable functions with square-integrable distributional derivatives, and $H_0^1(\Omega)$ is defined by

$$H_0^1(\Omega) = \{\psi \in H^1(\Omega);\ T(\psi) = 0\},$$

where $T : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the trace operator. Then the weak formulation of (2.3)-(2.4) can be defined in the usual way; Find $p \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla p)\, dx = \int_{\Omega_1} f\psi\, dx \quad \text{for all } \psi \in H_0^1(\Omega), \tag{3.3}$$

where the integral over Ω_1 on the right hand side of (3.3) is a consequence of assumption (2.5). If the boundary of Ω is sufficiently smooth, then it follows from (2.2), (3.1), (3.2) and the Lax-Milgram theorem that the problem (3.3) is well-posed for every $\delta > 0$, cf. e.g. Dautray and Lions [10].

Next, we make the following assumptions for the subdomains Ω_1 and Ω_δ; $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$, $\Omega_1 \cap \Omega_\delta = \emptyset$, $\partial\Omega \cap \overline{\Omega}_\delta = \emptyset$. That is, the closure $\overline{\Omega}_\delta$ of Ω_δ is contained in Ω and $\partial\Omega \subset \partial\Omega_1$, $\partial\Omega_\delta \subset \partial\Omega_1$ and $\partial\Omega \cup \partial\Omega_\delta = \partial\Omega_1$, see Figure 2.1. We will also assume that Ω_1 and Ω_δ have sufficiently smooth boundaries.

The Ritz-Galerkin discretization of the problem (3.3) is defined in the usual way. Let $\{N_1, \ldots, N_q\}$ be a set of linearly independent functions satisfying $N_i \in H_0^1(\Omega)$ for $i = 1, \ldots, q$, and define

$$V_h = \text{span}\,\{N_1, \ldots, N_q\}\,.$$

Here, h represents the global mesh parameter associated with the finite element space V_h. Clearly, V_h is a subspace of $H_0^1(\Omega)$ and we can define the finite dimensional approximation of (3.3) as follows; Find $p_h \in V_h$ such that

$$\int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla p_h)\, dx = \int_{\Omega_1} f\psi\, dx \quad \text{for all } \psi \in V_h. \tag{3.4}$$

In order to prove an error estimate of the form (2.6) we must introduce two assumptions on the finite element space V_h. To this end, consider the spaces

$$\begin{aligned} V_{\Omega_\delta,h} &= \{w_h|_{\Omega_\delta};\ w_h \in V_h\}\,, \\ V_{\Omega_1,h} &= \{w_h|_{\Omega_1};\ w_h \in V_h\}\,, \end{aligned} \tag{3.5}$$

and let $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$ denote the trace operator. The assumptions are now stated in terms of the following set

$$G_{\Omega_1,h} = \{T_{\Omega_1}(\psi)|_{\partial\Omega_\delta};\ \psi \in V_{\Omega_1,h}\},$$

which is well defined since $\partial\Omega_\delta \subset \partial\Omega_1$, cf. Figure 2.1.

A1. For every $w_h \in G_{\Omega_1,h}$ we assume that the following problem has a unique solution: Find $u_h \in V_{\Omega_\delta,h}$ such that $u_h = w_h$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla u_h)\, dx = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta). \tag{3.6}$$

We will also assume that there exists a constant c_1, not depending on δ or h, such that the solution u_h of this problem satisfies

$$\|u_h\|_{H^1(\Omega_\delta)} \le c_1 \|w_h\|_{H^{1/2}(\partial\Omega_\delta)}.$$

A2. If $\varphi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta)$ then the function

$$\psi = \begin{cases} \varphi & \text{on } \Omega_\delta \\ 0 & \text{on } \Omega_1 \end{cases}$$

belongs to V_h.

To motivate assumption **A1** we consider a similar feature in the continuous case. Let u be the solution of the following problem: Find $u \in H^1(\Omega_\delta)$ such that $u = w \in H^{1/2}(\partial\Omega_\delta)$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla u)\, dx = 0 \quad \text{for all } \psi \in H_0^1(\Omega_\delta).$$

Then it is well-known that u satisfies an inequality of the form

$$\|u\|_{H^1(\Omega_\delta)} \le C\|w\|_{H^{1/2}(\partial\Omega_\delta)},$$

where $C \in \mathbb{R}_+$ is a constant independent of δ, see e.g. Hackbusch [19]. Motivated by this property, which is valid in the continuous case, we will assume that **A1** holds throughout this paper. In fact, assumption **A1** can be verified for various types of finite element spaces, see Bramble, Pasciak and Schatz [2] and [3].

Next, condition **A2** makes it possible to extend discrete test functions defined on Ω_δ, and that vanish on $\partial\Omega_\delta$, to the entire domain Ω. Typically, this assumption is satisfied if the interface $\partial\Omega_\delta$ coincides with grid-lines of the mesh associated with the finite element space V_h.

4. An error estimate for general finite element approximations. In this section we want to prove an error estimate of the form (2.6). The main idea of our error analysis is to show that the best approximation τ_h, measured in a proper norm, of p and the finite element approximation p_h of p belong to a particular subspace of the finite element space V_h. More precisely, we will show that τ_h and p_h are so-called discrete Λ-harmonic functions in Ω_δ. Then we use assumption **A1** to prove that the error $p - p_h$ on Ω_δ is bounded by the error $p - p_h$ on Ω_1 and by the best approximation error $p - \tau_h$. Since $\Lambda_\delta(x) = \Lambda(x)$ for all $x \in \Omega_1$, cf. (2.2), it turns out that the error $p - p_h$ on Ω_1 can be estimated independently of δ.

For $\delta \in (0, 1]$ we define the inner-product $[\cdot,\cdot]_\delta$ on $H_0^1(\Omega)$ by

$$\begin{aligned} [\varphi,\psi]_\delta &= \int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla\varphi)\, dx \\ &= \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx + \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx \quad \text{for } \varphi,\psi \in H_0^1(\Omega), \end{aligned} \tag{4.1}$$

see (2.2). The associated energy norm is

$$\|\psi\|_\delta = \sqrt{[\psi,\psi]_\delta} \quad \text{for } \psi \in H_0^1(\Omega). \tag{4.2}$$

It is well-known that the solution p_h of (3.4) is the best approximation in V_h of p measured in the energy norm, i.e.

$$\|p - p_h\|_\delta = \inf_{q_h \in V_h} \|p - q_h\|_\delta, \tag{4.3}$$

see for instance [4]. Clearly, if δ is close to zero then the last term in (4.1) becomes very small. That is, the energy norm is very weak for small values of δ. Thus, a small error measured in the energy norm does not necessarily imply that p_h is a good approximation of p. This observation is our main motivation for measuring the error in a norm not depending on δ, namely in the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$.

Throughout the paper $[\cdot,\cdot]_1$ and $\|\cdot\|_1$ denote the inner-product and norm defined in (4.1) and (4.2) by putting $\delta = 1$. For easy reference, we now state some trivial properties of the $\|\cdot\|_1$, $\|\cdot\|_\delta$ and $\|\cdot\|_{H^1(\Omega)}$ norms. Since we assume that $0 < \delta \ll 1$ it is easy to verify that

$$\|\psi\|_\delta \leq \|\psi\|_1 \quad \text{for all } \psi \in H_0^1(\Omega), \tag{4.4}$$

see (4.1). Clearly, this inequality and equation (4.3) leads to the upper bound

$$\|p - p_h\|_\delta \leq \inf_{q_h \in V_h} \|p - q_h\|_1 \tag{4.5}$$

for the error. Furthermore, by Poincaré's inequality and inequality (3.2) it follows that the $\|\cdot\|_{H^1(\Omega)}$- and the $\|\cdot\|_1$-norm are equivalent independent of δ. That is, there exist constants c_2 and c_3, not depending on δ, such that

$$c_2\|\psi\|_{H^1(\Omega)} \leq \|\psi\|_1 \leq c_3\|\psi\|_{H^1(\Omega)} \quad \text{for all } \psi \in H_0^1(\Omega). \tag{4.6}$$

As explained above, the starting point of our error analysis is to prove that p_h and the best approximation τ_h, with respect to the $\|\cdot\|_1$-norm, of p belongs to a particular subspace of the finite element space V_h. To this end, consider the following subspaces of $H_0^1(\Omega)$ and V_h

$$\begin{aligned}
S_\delta &= \{\psi \in H_0^1(\Omega);\ \text{supp}(\psi) \subset \overline{\Omega}_\delta \text{ and } \psi|_{\Omega_\delta} \in H_0^1(\Omega_\delta)\},\\
S_1 &= S_\delta^\perp \text{ with respect to the } [\cdot,\cdot]_1\text{-inner-product},\\
S_{\delta,h} &= \{\psi \in V_h;\ \text{supp}(\psi) \subset \overline{\Omega}_\delta \text{ and } \psi|_{\Omega_\delta} \in H_0^1(\Omega_\delta)\},\\
S_{1,h} &= S_{\delta,h}^\perp \text{ with respect to the } [\cdot,\cdot]_1\text{-inner-product}.
\end{aligned} \tag{4.7}$$

That is, the functions in S_1 and $S_{1,h}$ are so-called Λ-harmonic and discrete Λ-harmonic functions in Ω_δ, respectively. In particular, if $q_h \in S_{1,h}$ then by the definition (4.7) of $S_{1,h}$ and $S_{\delta,h}$

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla q_h)\, dx = 0 \quad \text{for all } \psi \in S_{\delta,h},$$

and it follows from assumption **A2** that

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla q_h)\, dx = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta).$$

That is, $q_h|_{\Omega_\delta}$ solves a problem of the form (3.6) with $w_h = T_{\Omega_1}(q_h)|_{\partial\Omega_\delta}$. Hence, by assumption **A1** we conclude that

$$\|q_h\|_{H^1(\Omega_\delta)} \leq c_1 \|q_h\|_{H^{1/2}(\partial\Omega_\delta)} \quad \text{for all } q_h \in S_{1,h}.$$

LEMMA 4.1. *If assumptions* **A1** *and* **A2** *hold then there exists a constant* c_1, *independent of* δ *and* h, *such that*

$$\| q_h \|_{H^1(\Omega_\delta)} \leq c_1 \| q_h \|_{H^{1/2}(\partial\Omega_\delta)} \quad \textit{for all } q_h \in S_{1,h}, \tag{4.8}$$

where $S_{1,h}$ *is the function space defined in (4.7).*

Next, we prove that the solutions p and p_h of (3.3) and (3.4) are Λ-harmonic and discrete Λ-harmonic functions in Ω_δ, respectively.

LEMMA 4.2. *Let* p *and* p_h *be the solutions of (3.3) and (3.4), respectively. Assume that* f *satisfies (2.5) and that assumptions* **A1** *and* **A2** *hold, then*
a) $p \in S_1$ *and* $p_h \in S_{1,h}$.
b) *The* $[\cdot,\cdot]_1$*-projection* τ_h *of* p *in* V_h *belongs to* $S_{1,h}$, *i.e.* $\tau_h \in S_{1,h}$.

Proof. **Part a)** Let $\psi \in S_\delta$ be arbitrary, then $\mathrm{supp}(\psi) \subset \Omega_\delta$ and (3.3), (2.2) and assumption (2.5) imply that

$$\delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla p)\, dx = 0.$$

Hence, $[\psi, p]_1 = 0$ for all $\psi \in S_\delta$ and we conclude that $p \in S_1$. By a similar argument it is easy to verify that $p_h \in S_{1,h}$.

Part b) By the definition of τ_h

$$[\tau_h, \psi]_1 = [p, \psi]_1 \quad \text{for all } \psi \in V_h.$$

Since $p \in S_1$, cf. part a), and $S_{\delta,h} \subset S_\delta$ it follows that

$$[\tau_h, \psi]_1 = [p, \psi]_1 = 0 \quad \text{for all } \psi \in S_{\delta,h},$$

and we conclude that $\tau_h \in S_{1,h}$. □

With this information at hand, we are ready to prove our first error estimate.

THEOREM 4.3. *Suppose* f *satisfies (2.5) and that assumptions* **A1** *and* **A2** *hold. Then the finite element approximation* p_h *of* p *satisfies*

$$\|p - p_h\|_{H^1(\Omega)} \leq c \inf_{q_h \in V_h} \|p - q_h\|_{H^1(\Omega)}, \tag{4.9}$$

where c *is a constant independent of* δ *and* h. *Here* p *and* p_h *are the solutions of (3.3) and (3.4), respectively.*

Proof. First we prove that the error $p - p_h$ on Ω_δ can be bounded by the best approximation error $p - \tau_h$, measured in the $\|\cdot\|_1$-norm, and by the error $p - p_h$ on Ω_1. Let τ_h denote the $[\cdot,\cdot]_1$-projection of p in V_h, i.e.

$$\|p - \tau_h\|_1 = \inf_{q_h \in V_h} \|p - q_h\|_1. \tag{4.10}$$

Recall Lemma 4.2, that $\tau_h, p_h \in S_{1,h}$ and hence $\tau_h - p_h \in S_{1,h}$. Consequently, it follows from Lemma 4.1 that

$$\|p_h - \tau_h\|_{H^1(\Omega_\delta)} \leq c_1 \|p_h - \tau_h\|_{H^{1/2}(\partial\Omega_\delta)}. \tag{4.11}$$

The triangle inequality and inequalities (3.2) and (4.11) imply that

$$\left(\int_{\Omega_\delta} \nabla(p - p_h) \cdot (\Lambda \nabla (p - p_h))\, dx \right)^{1/2}$$

$$\le \left(\int_{\Omega_\delta} \nabla(p-\tau_h)\cdot(\Lambda\nabla(p-\tau_h))\,dx\right)^{1/2} + \left(\int_{\Omega_\delta} \nabla(\tau_h-p_h)\cdot(\Lambda\nabla(\tau_h-p_h))\,dx\right)^{1/2}$$
$$\le \|p-\tau_h\|_1 + \sqrt{M}\|p_h-\tau_h\|_{H^1(\Omega_\delta)}$$
$$\le \|p-\tau_h\|_1 + c_1\sqrt{M}\|p_h-\tau_h\|_{H^{1/2}(\partial\Omega_\delta)}$$
$$\le \|p-\tau_h\|_1 + c_1\sqrt{M}\|p-p_h\|_{H^{1/2}(\partial\Omega_\delta)} + c_1\sqrt{M}\|p-\tau_h\|_{H^{1/2}(\partial\Omega_\delta)}.$$

Now, recall that $\partial\Omega_\delta \subset \partial\Omega_1$, cf. Figure 2.1. Thus, from the trace theorem, the observation that $\Omega_1 \subset \Omega$ and inequalities (4.6) and (4.10) we find that

$$\left(\int_{\Omega_\delta} \nabla(p-p_h)\cdot(\Lambda\nabla(p-p_h))\,dx\right)^{1/2}$$
$$\le \|p-\tau_h\|_1 + c_1\sqrt{M}\|T_{\Omega_1}\|\,\|p-\tau_h\|_{H^1(\Omega_1)} + c_1\sqrt{M}\|T_{\Omega_1}\|\,\|p-p_h\|_{H^1(\Omega_1)}$$
$$\le \|p-\tau_h\|_1 + c_1\sqrt{M}\|T_{\Omega_1}\|\,\|p-\tau_h\|_{H^1(\Omega)} + c_1\sqrt{M}\|T_{\Omega_1}\|\,\|p-p_h\|_{H^1(\Omega_1)}$$
$$\le (1+\sqrt{M}c_1/c_2\,\|T_{\Omega_1}\|)\inf_{q_h\in V_h}\|p-q_h\|_1 + c_1\sqrt{M}\|T_{\Omega_1}\|\,\|p-p_h\|_{H^1(\Omega_1)}, \tag{4.12}$$

where $\|T_{\Omega_1}\|$ denotes the operator norm of the trace operator $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$.

Next, we want to prove that the Sobolev norm of $p - p_h$ on Ω_1 is bounded independently of δ. From assumption (3.2), the definition (4.1)-(4.2) of the $\|\cdot\|_\delta$-norm and inequality (4.5) it follows that

$$m\int_{\Omega_1} \nabla(p-p_h)\cdot\nabla(p-p_h)\,dx \le \int_{\Omega_1} \nabla(p-p_h)\cdot(\Lambda\nabla(p-p_h))\,dx$$
$$\le \|p-p_h\|_\delta^2 \le \inf_{q_h\in V_h}\|p-q_h\|_1^2. \tag{4.13}$$

Clearly, by this inequality and Poincaré's inequality there exists a constant c_4, independent of δ and h, such that

$$\|p-p_h\|_{H^1(\Omega_1)} \le c_4 \inf_{q_h\in V_h}\|p-q_h\|_1. \tag{4.14}$$

Combining (4.12) and (4.14) we obtain the following inequality

$$\int_{\Omega_\delta} \nabla(p-p_h)\cdot(\Lambda\nabla(p-p_h))\,dx \le c_5 \inf_{q_h\in V_h}\|p-q_h\|_1^2, \tag{4.15}$$

where c_5 is a constant not depending on h or δ. Finally, from (4.13) and (4.15) we find that

$$\|p-p_h\|_1 \le \sqrt{1+c_5}\inf_{q_h\in V_h}\|p-q_h\|_1, \tag{4.16}$$

and the theorem follows from (4.6). □

5. Regularity. From (2.2), (3.1), (3.2), (3.3) and the Lax-Milgram theorem it follows that the solution p of (3.3) belongs to the Sobolev space $H_0^1(\Omega)$ for all $\delta \in (0,1]$. It is well-known that this property of p is not sufficient in order to prove an error estimate of the form (2.7). We need more information about the regularity of p, cf. e.g. Hackbusch [19]. Recall that Λ_δ has a jump discontinuity at the interface $\partial\Omega_\delta$, cf. (2.2). Hence, even if the mobility tensor $\Lambda = (\lambda_{i,j})$ satisfies

$$\lambda_{i,j} \in C^1(\overline{\Omega}) \quad \text{for } i,j = 1,2, \tag{5.1}$$

p will in general not belong to $H^2(\Omega)$. However, we will show that if the boundaries of the subdomains Ω_1 and Ω_δ are sufficiently smooth and (5.1) holds, then the solution

p of (3.3) is piecewise smooth, i.e. $p|_{\Omega_\delta} \in H^2(\Omega_\delta)$ and $p|_{\Omega_1} \in H^2(\Omega_1)$. Moreover, we prove that the associated Sobolev norms $\|p\|_{H^2(\Omega_\delta)}$ and $\|p\|_{H^2(\Omega_1)}$ are bounded independently of δ.

As mentioned above, interface problems similar to problems of the form (2.2)-(2.4) have been studied by several authors. In fact, it is well-known that if (5.1) holds and the boundaries of Ω_1 and Ω_δ are smooth, then the solution p of a problem of this form is piecewise smooth, cf. Hackbusch [19, Ch.10] and Ladyzhenskaya [22, Ch.V]. However, to our knowledge it is not known that the associated Sobolev norms are bounded independently of δ, provided that condition (2.5) is satisfied. This latter observation is our main motivation for presenting the following analysis.

The main idea of the regularity proof in this section is to construct a sequence of piecewise smooth functions that converge, in proper norms, to the solution p of (3.3). This sequence is constructed by solving appropriate elliptic boundary value problems on Ω_1 and Ω_δ.

In the rest of this paper we assume that (5.1) is satisfied and that the following assumption holds.

A3. For every $g \in H^{1/2}(\partial\Omega_\delta)$ we assume that the weak solution u of the following problem

$$
\begin{aligned}
\nabla \cdot (\Lambda \nabla u) &= -f \quad \text{in } \Omega_1, \\
u &= 0 \quad \text{on } \partial\Omega, \\
(\Lambda \nabla u) \cdot \mathbf{n}_1 &= -g \quad \text{on } \partial\Omega_\delta
\end{aligned}
\tag{5.2}
$$

belongs to $H^2(\Omega_1)$, i.e. $u \in H^2(\Omega_1)$ and that an inequality of the form

$$
\|u\|_{H^2(\Omega_1)} \leq c_6(\|f\|_{L^2(\Omega_1)} + \|g\|_{H^{1/2}(\partial\Omega_\delta)} + \|u\|_{H^1(\Omega_1)})
\tag{5.3}
$$

holds. Here, c_6 is a constant independent of δ. We will also assume that the Λ-harmonic extension $\widetilde{u}$ of $T_{\Omega_1}(u)|_{\partial\Omega_\delta}$ to Ω_δ belongs to $H^2(\Omega_\delta)$. More precisely, we assume that the weak solution $\widetilde{u}$ of the problem

$$
\begin{aligned}
&\widetilde{u} = T_{\Omega_1}(u)|_{\partial\Omega_\delta} \in H^{3/2}(\partial\Omega_\delta) \quad \text{on } \partial\Omega_\delta, \\
&\nabla \cdot (\Lambda \nabla \widetilde{u}) = 0 \quad \text{in } \Omega_\delta,
\end{aligned}
\tag{5.4}
$$

satisfies $\widetilde{u} \in H^2(\Omega_\delta)$, and that there exists a constant c_7 independent of δ and h, such that

$$
\|\widetilde{u}\|_{H^2(\Omega_\delta)} \leq c_7(\|T_{\Omega_1}(u)\|_{H^{3/2}(\partial\Omega_\delta)} + \|\widetilde{u}\|_{H^1(\Omega_\delta)}).
\tag{5.5}
$$

If the boundaries of Ω_δ and Ω_1 are sufficiently smooth then **A3** holds, cf. e.g. Hackbusch [19, Ch.9]. Moreover, it is well-known that the weak solution $\widetilde{u}$ of (5.4) satisfies an inequality of the form

$$
\|\widetilde{u}\|_{H^1(\Omega_\delta)} \leq c_8(\|T_{\Omega_1}(u)\|_{H^{1/2}(\partial\Omega_\delta)}) \leq c_8\|T_{\Omega_1}\|_1 \, \|u\|_{H^1(\Omega_1)},
\tag{5.6}
$$

where the last inequality is a consequence of the trace theorem. Combining inequalities (5.5) and (5.6) we find that

$$
\begin{aligned}
\|\widetilde{u}\|_{H^2(\Omega_\delta)} &\leq c_7(\|T_{\Omega_1}(u)\|_{H^{3/2}(\partial\Omega_\delta)} + c_8\|T_{\Omega_1}\|_1 \, \|u\|_{H^1(\Omega_1)}) \\
&\leq c_7(\|T_{\Omega_1}\|_2 \, \|u\|_{H^2(\Omega_1)} + c_8\|T_{\Omega_1}\|_1 \, \|u\|_{H^1(\Omega_1)}) \\
&\leq c_9\|u\|_{H^2(\Omega_1)},
\end{aligned}
\tag{5.7}
$$

where c_9 does not depend on δ or h. Here, $\|T_{\Omega_1}\|_1$ and $\|T_{\Omega_1}\|_2$ denotes the operator norms of the trace operators $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$ and $T_{\Omega_1} : H^2(\Omega_1) \to H^{3/2}(\partial\Omega_1)$, respectively. These operator norms are independent of δ and h, and in the rest of this paper we will simply write $\|T_{\Omega_1}\|$ whenever we need to refer to the $\|T_{\Omega_1}\|_1$- or the $\|T_{\Omega_1}\|_2$-norm.

In order to define a sequence of piecewise smooth approximations of the solution p of (3.3) we must rewrite problems (5.2) and (5.4) on a form more suitable for our analysis. To this end, let $\mathcal{P} : H_0^1(\Omega) \to S_\delta$ denote the projection operator on to S_δ with respect to the $[\cdot,\cdot]_1$-inner-product, where we recall that $[\cdot,\cdot]_1$ is the inner-product defined in (4.1) by putting $\delta = 1$. Next, consider the bilinear form $b(\cdot,\cdot)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ as follows

$$b(\psi,\varphi) = \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx + \int_{\Omega_\delta} \nabla\mathcal{P}(\psi) \cdot (\Lambda\nabla\mathcal{P}(\varphi))\, dx. \tag{5.8}$$

Now it turns out that problems (5.2) and (5.4) can be solved by solving a problem of the form; Find $v \in H_0^1(\Omega)$ such that

$$b(\psi, v) = \int_{\Omega_1} f\psi\, dx - \int_{\partial\Omega_\delta} g\psi\, ds \quad \text{for all } \psi \in H_0^1(\Omega). \tag{5.9}$$

LEMMA 5.1. *If v solves (5.9) then $u = v|_{\Omega_1}$ and $\tilde{u} = v|_{\Omega_\delta}$ are the weak solutions of (5.2) and (5.4), respectively. Furthermore, if u and $\tilde{u}$ are the weak solutions of (5.2) and (5.4), then*

$$v = \begin{cases} u & \text{in } \Omega_1, \\ \tilde{u} & \text{in } \Omega_\delta, \end{cases}$$

solves (5.9).

Proof. Assume that v solves (5.9). Let $\psi \in S_\delta$ be arbitrary, then $\mathcal{P}(\psi) = \psi$ and $\mathrm{supp}(\psi) \subset \Omega_\delta$. From the definition (5.8) of $b(\cdot,\cdot)$ and (5.9) we find that

$$\begin{aligned} 0 &= \int_{\Omega_\delta} \nabla\mathcal{P}(\psi) \cdot (\Lambda\nabla\mathcal{P}(v))\, dx = [\psi, \mathcal{P}(v)]_1 = [\psi, \mathcal{P}(v) - v]_1 + [\psi, v]_1 \\ &= 0 + [\psi, v]_1 = \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla v)\, dx \quad \text{for all } \psi \in S_\delta. \end{aligned} \tag{5.10}$$

Hence, from (5.10) it follows that $v \in S_1$ and therefore $\mathcal{P}(v) = 0$. Thus, (5.9) implies that

$$\int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla v)\, dx = \int_{\Omega_1} f\psi\, dx - \int_{\partial\Omega_\delta} g\psi\, ds \quad \text{for all } \psi \in H_0^1(\Omega).$$

Since every $\widehat{\psi} \in W = \{\varphi \in H^1(\Omega_1);\ \varphi = 0 \text{ on } \partial\Omega\}$ can be extended to a function $\psi \in H_0^1(\Omega)$ such that $\psi = \widehat{\psi}$ on Ω_1 we conclude that

$$\int_{\Omega_1} \nabla\widehat{\psi} \cdot (\Lambda\nabla v)\, dx = \int_{\Omega_1} f\widehat{\psi}\, dx - \int_{\partial\Omega_\delta} g\widehat{\psi}\, ds \quad \text{for all } \widehat{\psi} \in W.$$

Thus, $u = v|_{\Omega_1}$ is the weak solution of (5.2). Next, since every $\overline{\psi} \in H_0^1(\Omega_\delta)$ has a canonical extension $\psi \in S_\delta \subset H_0^1(\Omega)$, defined by putting $\psi = \overline{\psi}$ in Ω_δ and $\psi = 0$ in Ω_1, it follows from (5.10) that v satisfies an equation of the form

$$\int_{\Omega_\delta} \nabla\overline{\psi} \cdot (\Lambda\nabla v)\, dx = 0 \quad \text{for all } \overline{\psi} \in H_0^1(\Omega_\delta).$$

That is, $\tilde{u} = v|_{\Omega_\delta}$ is the weak solution of (5.4).

Now, let u and $\tilde{u}$ be the weak solutions of (5.2) and (5.4), respectively. Then it follows immediately that the function v defined by

$$v = \begin{cases} u & \text{in } \Omega_1, \\ \tilde{u} & \text{in } \Omega_\delta \end{cases}$$

belongs to S_1, and that $\mathcal{P}(v) = 0$. Therefore, since u satisfies an equation of the form

$$\int_{\Omega_1} \nabla\widehat{\psi}\cdot(\Lambda\nabla u)\,dx = \int_{\Omega_1} f\widehat{\psi}\,dx - \int_{\partial\Omega_\delta} g\widehat{\psi}\,ds \quad \text{for all } \widehat{\psi} \in \{\varphi \in H^1(\Omega_1);\ \varphi = 0 \text{ on } \partial\Omega\},$$

it follows that v must solve (5.9). □

The next result is a corollary of assumption **A3**, Lemma 5.1 and its proof. The proof of Lemma 5.2 is straight forward and therefore omitted.

LEMMA 5.2. *For every $g \in H^{1/2}(\partial\Omega_\delta)$ the solution v of (5.9) satisfies*
a) $v \in S_1$, $v|_{\Omega_1} \in H^2(\Omega_1)$ *and* $v|_{\Omega_\delta} \in H^2(\Omega_\delta)$.
b)

$$\int_{\Omega_\delta} \psi\nabla\cdot(\Lambda\nabla v)\,dx = 0 \quad \textit{for all } \psi \in L^2(\Omega_\delta). \tag{5.11}$$

Now we are ready to construct a sequence of piecewise smooth approximations of the solution p of (3.3). First, let $a(\cdot,\cdot)$ be the bilinear form associated with the problem (3.3), i.e.

$$a(\psi,\varphi) = \int_{\Omega_1} \nabla\psi\cdot(\Lambda\nabla\varphi)\,dx + \delta\int_{\Omega_\delta} \nabla\psi\cdot(\Lambda\nabla\varphi)\,dx.$$

Inspired by Koshelev [21], the sequence $\{p^{(n)}\}_{n=0}^{\infty}$ of approximations of p is defined recursively as follows: find $p^{(n)} \in H_0^1(\Omega)$ such that

$$b(\psi,p^{(n)}) - b(\psi,p^{(n-1)}) + a(\psi,p^{(n-1)}) = \int_{\Omega_1} f\psi\,dx \quad \text{for all } \psi \in H_0^1(\Omega). \tag{5.12}$$

Here, $p^{(0)}$ is the solution of the following problem: find $p^{(0)} \in H_0^1(\Omega)$ such that

$$b(\psi,p^{(0)}) = \int_{\Omega_1} f\psi\,dx \quad \text{for all } \psi \in H_0^1(\Omega). \tag{5.13}$$

Notice that if $p^{(n)}$ has a limit $\overline{q}$ in $H_0^1(\Omega)$, then it follows immediately from (5.12) that $\overline{q}$ solves (3.3), i.e. by the uniqueness of the solution $\overline{q} = p$.

Let us verify that the functions $\{p^{(n)}\}_{n=0}^{\infty}$ are Λ-harmonic in Ω_δ and piecewise smooth in Ω_1 and Ω_δ.

LEMMA 5.3. *Let $\{p^{(n)}\}_{n=0}^{\infty}$ be the sequence of functions defined in (5.12) and (5.13). If assumption* **A3** *holds then*

$$p^{(n)} \in S_1,\ \ p^{(n)}|_{\Omega_1} \in H^2(\Omega_1) \textit{ and } p^{(n)}|_{\Omega_\delta} \in H^2(\Omega_\delta) \quad \textit{for } n = 0,1,2,\ldots. \tag{5.14}$$

Furthermore, $p^{(n)}$ satisfies an equation of the form

$$b(\psi,p^{(n)}) = \int_{\Omega_1} f\psi\,dx + \int_{\partial\Omega_\delta} g_n\psi\,ds \quad \textit{for all } \psi \in H_0^1(\Omega), \tag{5.15}$$

where $g_0 = 0$,

$$g_n = \delta T_{\Omega_\delta}(\Lambda\nabla p^{(n-1)})\cdot\mathbf{n}_1 \quad \textit{for } n = 1,2,3,\ldots.$$

and[2] $\mathbf{n}_1$ *represents the outer unit normal vector to Ω_1.*

Proof. From the definition (5.13) of $p^{(0)}$ and Lemma 5.2 it follows that (5.14) and (5.15) hold for $n = 0$ and $g_0 = 0$. Next, assume that (5.14) and (5.15) hold for

[2] Here, $T_{\Omega_\delta} : H^1(\Omega_\delta) \to H^{1/2}(\partial\Omega_\delta)$ denotes the trace operator, and for every $\mathbf{w} = (w_1,w_2) \in H_0^1(\Omega)\times H_0^1(\Omega)$ we define $T_{\Omega_\delta}(\mathbf{w})$ by $T_{\Omega_\delta}(\mathbf{w}) = (T_{\Omega_\delta}(w_1), T_{\Omega_\delta}(w_2))$

$0, 1, 2, \ldots, n-1$. By the definition (5.12) of $p^{(n)}$ and the assumption that $p^{(n-1)} \in S_1$ we find that

$$\begin{aligned} b(\psi, p^{(n)}) &= \int_{\Omega_1} f\psi \, dx - a(\psi, p^{(n-1)}) + b(\psi, p^{(n-1)}) \\ &= \int_{\Omega_1} f\psi \, dx - \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla p^{(n-1)}) \, dx - \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla p^{(n-1)}) \, dx \\ &\quad + \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla p^{(n-1)}) \, dx + \int_{\Omega_\delta} \nabla\mathcal{P}(\psi) \cdot (\Lambda \nabla \mathcal{P}(p^{(n-1)})) \, dx \\ &= \int_{\Omega_1} f\psi \, dx - \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla p^{(n-1)}) \, dx \quad \text{for all } \psi \in H_0^1(\Omega). \end{aligned}$$

Hence, since $p^{(n-1)}$ satisfies (5.15) it follows from Green's theorem and Lemma 5.2 that

$$\begin{aligned} b(\psi, p^{(n)}) &= \int_{\Omega_1} f\psi \, dx - \delta \int_{\partial\Omega_\delta} \psi \, T_{\Omega_\delta}(\Lambda \nabla p^{(n-1)}) \cdot \mathbf{n}_\delta \, ds + \delta \int_{\Omega_\delta} \psi \nabla \cdot (\Lambda \nabla p^{(n-1)}) \, dx \\ &= \int_{\Omega_1} f\psi \, dx + \int_{\partial\Omega_\delta} \psi \, \delta T_{\Omega_\delta}(\Lambda \nabla p^{(n-1)}) \cdot \mathbf{n}_1 \, ds \quad \text{for all } \psi \in H_0^1(\Omega), \end{aligned}$$

where $T_{\Omega_\delta}(\cdot)$ always represents the trace from Ω_δ (i.e. no assumption of equality of the traces from Ω_1 and Ω_δ is assumed). Here the last equality is a consequence of the assumption that $\partial\Omega_1 = \partial\Omega \cup \partial\Omega_\delta$ and that $\mathbf{n}_\delta = -\mathbf{n}_1$ on $\partial\Omega_\delta$, cf. Figure 2.1. Now, $p^{(n-1)}|_{\Omega_\delta} \in H^2(\Omega_\delta)$ and therefore

$$g_n = \delta T_{\Omega_\delta}(\Lambda \nabla p^{(n-1)}) \cdot \mathbf{n}_1 \tag{5.16}$$

belongs to $H^{1/2}(\partial\Omega_\delta)$, provided that the boundary $\partial\Omega_\delta$ of Ω_δ is sufficiently smooth. Hence, $p^{(n)}$ satisfies an equation of the form (5.15), and from Lemma 5.2 we conclude that $p^{(n)}|_{\Omega_1} \in H^2(\Omega_1)$ and that $p^{(n)}|_{\Omega_\delta} \in H^2(\Omega_\delta)$. Now, the lemma follows by induction. □

Next, we must prove that the sequence $\{p^{(n)}\}_{n=0}^{\infty}$ defined in (5.12) and (5.13) converge to p in $H_0^1(\Omega)$. To this end, let us have a closer look at the function space S_1 defined in (4.7). Recall that the functions in S_1 are so-called Λ-harmonic functions in Ω_δ, i.e. if $q \in S_1$ then

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla q) \, dx = 0 \quad \text{for all } \psi \in S_\delta.$$

Thus, it follows that $\widetilde{q} = q|_{\Omega_\delta}$ solves a problem of the form: Find $\widetilde{q} \in H^1(\Omega_\delta)$ such that $\widetilde{q} = T_{\Omega_1}(q)|_{\partial\Omega_\delta}$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla \widetilde{q}) \, dx = 0 \quad \text{for all } \psi \in H_0^1(\Omega_\delta). \tag{5.17}$$

Hence, we find that there exists a constant c_{10}, independent of δ, such that

$$\|\widetilde{q}\|_{H^1(\Omega_\delta)} \leq c_{10} \|T_{\Omega_1}(q)\|_{H^{1/2}(\partial\Omega_\delta)},$$

cf. e.g. Hackbusch [19]. By a trace inequality we conclude that

$$\|q\|_{H^1(\Omega_\delta)} \leq c_{11} \|q\|_{H^1(\Omega_1)} \quad \text{for all } \psi \in S_1,$$

where c_{11} does not depend on δ or h.

LEMMA 5.4. *There exists a constant c_{11}, independent of δ and h, such that*

$$\|q\|_{H^1(\Omega_\delta)} \leq c_{11} \|q\|_{H^1(\Omega_1)} \quad \textit{for all } \psi \in S_1. \tag{5.18}$$

Here, S_1 is the function space defined in (4.7).

With this information at hand, we are ready to prove that $p^{(n)}$ converge to p in $H_0^1(\Omega)$.

LEMMA 5.5. *Let p be the solution of the problem (3.3) and assume that f satisfies (2.5). Then there exists a positive constant δ_0 such that for every $\delta \in (0, \delta_0)$ the sequence $\{p^{(n)}\}_{n=0}^{\infty}$ defined in (5.12) and (5.13) converge to p in $H_0^1(\Omega)$.*

Proof. From (5.12) we find that

$$b(\psi, p^{(n)}) = \int_{\Omega_1} f\psi \, dx - a(\psi, p^{(n-1)}) + b(\psi, p^{(n-1)}) \quad \text{for all } \psi \in H_0^1(\Omega),$$

and

$$b(\psi, p^{(n-1)}) = \int_{\Omega_1} f\psi \, dx - a(\psi, p^{(n-2)}) + b(\psi, p^{(n-2)}) \quad \text{for all } \psi \in H_0^1(\Omega).$$

That is, the function $d^{(n)} = p^{(n)} - p^{(n-1)}$ satisfies an equation of the form

$$b(\psi, d^{(n)}) = b(\psi, d^{(n-1)}) - a(\psi, d^{(n-1)}) \quad \text{for all } \psi \in H_0^1(\Omega),$$

where $d^{(n-1)} = p^{(n-1)} - p^{(n-2)}$. Since $p^{(n)} \in S_1$ for all $n \in \mathbb{N}_0$, cf. Lemma 5.3, it follows that $d^{(n-1)}, d^{(n)} \in S_1$ and that $\mathcal{P}(d^{(n-1)}) = \mathcal{P}(d^{(n)}) = 0$. Consequently, we find that

$$\begin{aligned}\int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla d^{(n)}) \, dx &= \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla d^{(n-1)}) \, dx - \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla d^{(n-1)}) \, dx \\ &\quad -\delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla d^{(n-1)}) \, dx \\ &= -\delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla d^{(n-1)}) \, dx, \quad \text{for all } \psi \in H_0^1(\Omega).\end{aligned}$$

By putting $\psi = d^{(n)} \in H_0^1(\Omega)$ and applying (3.2) and Schwarz's inequality it follows that

$$m \int_{\Omega_1} |\nabla d^{(n)}|^2 \, dx \le M\delta \left(\int_{\Omega_\delta} |\nabla d^{(n-1)}|^2 \, dx \right)^{1/2} \left(\int_{\Omega_\delta} |\nabla d^{(n)}|^2 \, dx \right)^{1/2}.$$

Next, recall that $d^{(n-1)}, d^{(n)} \in S_1$. Hence, Lemma 5.4 and Poincaré's inequality imply that

$$\int_{\Omega_1} |\nabla d^{(n)}|^2 \, dx \le \delta c_{12} \left(\int_{\Omega_1} |\nabla d^{(n-1)}|^2 \, dx \right)^{1/2} \left(\int_{\Omega_1} |\nabla d^{(n)}|^2 \, dx \right)^{1/2},$$

where c_{12} is independent of δ. That is

$$\left(\int_{\Omega_1} |\nabla p^{(n)} - \nabla p^{(n-1)}|^2 \, dx \right)^{1/2} \le \delta c_{12} \left(\int_{\Omega_1} |\nabla p^{(n-1)} - \nabla p^{(n-2)}|^2 \, dx \right)^{1/2}.$$

Thus, if $\delta < 1/c_{12} = \delta_0$ then the sequence $\{p^{(n)}|_{\Omega_1}\}_{n=0}^{\infty}$ defines a contraction in the $\|\cdot\|_{\Omega_1}$-norm, defined as follows

$$\|\psi\|_{\Omega_1} = \left(\int_{\Omega_1} |\nabla\psi|^2 \, dx \right)^{1/2} \quad \text{for } \psi \in \{\varphi \in H^1(\Omega_1);\ \varphi = 0 \text{ on } \partial\Omega\}.$$

Therefore there exists a function $q \in \{\varphi \in H^1(\Omega_1);\ \varphi = 0 \text{ on } \partial\Omega\}$ such that $p^{(n)}|_{\Omega_1}$ converge to q in the $\|\cdot\|_{\Omega_1}$-norm. Clearly, Poincaré's inequality implies that $p^{(n)}|_{\Omega_1}$ also converge to q in the Sobolev norm $\|\cdot\|_{H^1(\Omega_1)}$.

Next, let $\widetilde{q}$ denote the Λ-harmonic extension of $T_{\Omega_1}(q)|_{\partial\Omega_\delta}$ to Ω_δ, i.e. $\widetilde{q}$ is the solution of the following problem: Find $\widetilde{q} \in H^1(\Omega_\delta)$ such that $\widetilde{q} = T_{\Omega_1}(q)|_{\partial\Omega_\delta}$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\widetilde{q})\, dx = 0 \quad \text{for all } \psi \in H_0^1(\Omega_\delta).$$

Then, the function

$$\overline{q}(x) = \begin{cases} q(x) & \text{for } x \in \Omega_1, \\ \widetilde{q}(x) & \text{for } x \in \Omega_\delta, \end{cases}$$

belongs to S_1, and from Lemma 5.3 and Lemma 5.4 we find that

$$\|p^{(n)} - \overline{q}\|_{H^1(\Omega_\delta)} \le c_{11}\|p^{(n)} - \overline{q}\|_{H^1(\Omega_1)} = c_{11}\|p^{(n)} - q\|_{H^1(\Omega_1)}.$$

Hence, since $p^{(n)}|_{\Omega_1}$ converge to q in $H^1(\Omega_1)$ we conclude that $p^{(n)}$ converge to $\overline{q}$ in $H_0^1(\Omega)$.

Finally, by taking limits in (5.12) it follows that

$$a(\psi, \overline{q}) = \int_{\Omega_1} f\psi\, dx \quad \text{for all } \psi \in H_0^1(\Omega),$$

and since the solution p of the problem (3.3) is unique we conclude that $\overline{q} = p$. □

In [24] we analysed several mathematical properties of problems of the form (3.3). In particular, we proved that the Sobolev norm $\|p\|_{H^1(\Omega)}$ of p is bounded independently of δ, i.e.

$$\|p\|_{H^1(\Omega)} \le C \text{ (independent of } \delta). \tag{5.19}$$

Thus, Lemma 5.5 implies that there exists a constant c_{13}, independent of $\delta \in (0, \delta_0)$, such that

$$\|p^{(n)}\|_{H^1(\Omega)} \le c_{13} \quad \text{for all } n \in \mathbb{N}_0. \tag{5.20}$$

This inequality and Lemma 5.1-Lemma 5.5 lead to the main result of this section.

PROPOSITION 5.6. *Assume that f satisfies (2.5) and that assumption* **A3** *holds. Then there exists a constant $\delta_0 \in \mathbb{R}_+$ such that for every $\delta \in (0, \delta_0)$ the solution p of (3.3) satisfies*

a) $p|_{\Omega_1} \in H^2(\Omega_1)$ *and* $\|p\|_{H^2(\Omega_1)} \le c$,

b) $p|_{\Omega_\delta} \in H^2(\Omega_\delta)$ *and* $\|p\|_{H^2(\Omega_\delta)} \le c$.

Here, c is a constant independent of $\delta \in (0, \delta_0)$.

Proof. Recall Lemma 5.3, that $p^{(n)} \in S_1$ and

$$b(\psi, p^{(n)}) = \int_{\Omega_1} f\psi\, dx + \delta \int_{\partial\Omega_\delta} \psi\, T_{\Omega_\delta}(\Lambda\nabla p^{(n-1)}) \cdot \mathbf{n}_1\, ds$$

for all $\psi \in H_0^1(\Omega)$, cf. equation (5.15). From Lemma 5.1 and assumption **A3** we find that

$$\|p^{(n)}\|_{H^2(\Omega_1)} \le c_6(\|f\|_{L^2(\Omega_1)} + \delta\|T_{\Omega_\delta}(\Lambda\nabla p^{(n-1)}) \cdot \mathbf{n}_1\|_{H^{1/2}(\partial\Omega_\delta)} + \|p^{(n)}\|_{H^1(\Omega_1)}).$$

Since the boundary $\partial\Omega_\delta$ of Ω_δ is smooth and Λ is assumed to satisfy (3.2) and (5.1) it follows that

$$\|p^{(n)}\|_{H^2(\Omega_1)} \le c_6(\|f\|_{L^2(\Omega_1)} + \delta c_{14}(\|T_{\Omega_\delta}(p^{(n-1)})\|_{H^{3/2}(\partial\Omega_\delta)} + \|p^{(n)}\|_{H^1(\Omega_1)}).$$

Next, the boundedness of the trace operator T_{Ω_δ}, and inequality (5.20) implies that there exists a constant c_{15}, independent of δ, such that

$$\begin{aligned}\|p^{(n)}\|_{H^2(\Omega_1)} &\le c_6(\|f\|_{L^2(\Omega_1)} + \delta c_{15}\|p^{(n-1)}\|_{H^2(\Omega_\delta)} + c_{13}) \\ &\le c_6(\|f\|_{L^2(\Omega_1)} + \delta c_{15}c_9\|p^{(n-1)}\|_{H^2(\Omega_1)} + c_{13}),\end{aligned}$$

where the last inequality follows from Lemma 5.3, Lemma 5.1 and inequality (5.7). That is

$$\|p^{(n)}\|_{H^2(\Omega_1)} \le c_{16}(\|f\|_{L^2(\Omega_1)} + \delta\|p^{(n-1)}\|_{H^2(\Omega_1)} + 1),$$

where $c_{16} = c_6 \max\{c_{15}c_9, c_{13}, 1\}$. By induction it follows that

$$\|p^{(n)}\|_{H^2(\Omega_1)} \le (\|f\|_{L^2(\Omega_1)} + 1)c_{16}\sum_{i=0}^{n-1}(c_{16}\delta)^i + (c_{16}\delta)^n\|p^{(0)}\|_{H^2(\Omega_1)}$$

for all $n \in \mathbb{N}$. Hence, if $0 < \delta < \delta_0 < 1/c_{16}$ then

$$\|p^{(n)}\|_{H^2(\Omega_1)} \le c_{17} \quad \text{for all } n \in \mathbb{N}, \tag{5.21}$$

where c_{17} does not depend on $\delta \in (0, \delta_0)$.

Next, inequality (5.21) implies that $\{(p^{(n)}|_{\Omega_1})_{xx}\}_{n=0}^{\infty}$ defines a uniformly bounded sequence of functions in $L^2(\Omega_1)$. Thus, there exists a subsequence $\{(p^{(n_j)}|_{\Omega_1})_{xx}\}_{j=0}^{\infty}$ and a function $q \in L^2(\Omega_1)$ such that $(p^{(n_j)}|_{\Omega_1})_{xx}$ converge weakly to q in $L^2(\Omega_1)$, i.e.

$$\lim_{j\to\infty}(p_{xx}^{(n_j)}, \varphi)_{L^2(\Omega_1)} = (q, \varphi)_{L^2(\Omega_1)} \quad \text{for all } \varphi \in L^2(\Omega_1), \tag{5.22}$$

cf. e.g. Griffel [17]. In particular,

$$\lim_{j\to\infty}(p_{xx}^{(n_j)}, \varphi)_{L^2(\Omega_1)} = (q, \varphi)_{L^2(\Omega_1)} \quad \text{for all } \varphi \in C_0^{\infty}(\Omega_1). \tag{5.23}$$

Now, since $p^{(n_j)}|_{\Omega_1} \in H^2(\Omega_1)$ for all $j \in \mathbb{N}$ and $p^{(n_j)}|_{\Omega_1}$ converge strongly to $p|_{\Omega_1}$ in $H^1(\Omega_1)$, cf. Lemma 5.5, it follows that

$$\lim_{j\to\infty}(p_{xx}^{(n_j)}, \varphi)_{L^2(\Omega_1)} = -\lim_{j\to\infty}(p_x^{(n_j)}, \varphi_x)_{L^2(\Omega_1)} = -(p_x, \varphi_x)_{L^2(\Omega_1)} \quad \text{for all } \varphi \in C_0^{\infty}(\Omega_1). \tag{5.24}$$

Hence, from (5.23) and (5.24) we conclude that

$$(q, \varphi)_{L^2(\Omega_1)} = -(p_x, \varphi_x)_{L^2(\Omega_1)} \quad \text{for all } \varphi \in C_0^{\infty}(\Omega_1),$$

and it follows that $(p|_{\Omega_1})_{xx} = q \in L^2(\Omega_1)$. Furthermore, putting $\varphi = (p|_{\Omega_1})_{xx} \in L^2(\Omega_1)$ in (5.22) and applying (5.21) and Schwarz's inequality yield

$$|(p_{xx}, p_{xx})_{L^2(\Omega_1)}| = \lim_{j\to\infty}|(p_{xx}^{(n_j)}, p_{xx})_{L^2(\Omega_1)}| \le \sup_j \|p_{xx}^{(n_j)}\|_{L^2(\Omega_1)}\|p_{xx}\|_{L^2(\Omega_1)} \le c_{17}\|p_{xx}\|_{L^2(\Omega_1)},$$

where we recall that c_{17} is independent of δ. Thus,

$$\|p_{xx}\|_{L^2(\Omega_1)} \le c_{17}. \tag{5.25}$$

In a similar manner it can be verified that $(p|_{\Omega_1})_{yy}$, $(p|_{\Omega_1})_{xy} \in L^2(\Omega_1)$, and that the associated L^2-norms are bounded independently of δ. Hence, we conclude that part **a)** of the proposition must hold.

From Lemma 5.3, Lemma 5.1, assumption **A3** and inequalities (5.7) and (5.21) we find that

$$\|p^{(n)}\|_{H^2(\Omega_\delta)} \le c_9\|p^{(n)}\|_{H^2(\Omega_1)} \le c_9\, c_{17} \quad \text{for all } n \in \mathbb{N}.$$

The rest of the proof of part **b)** is analogous to the proof of part **a)** and therefore omitted, cf. equations (5.21)-(5.25). □

By Proposition 5.6 there exists a constant δ_0 such that if $0 < \delta < \delta_0$ then the solution p of (3.3) is piecewise smooth. Furthermore, the associated Sobolev norms are bounded independently of δ. What happens if $\delta > \delta_0$? As mentioned in the introduction of this section, it is well-known that the solution p of an interface problem of the form (2.2)-(2.4) is smooth in Ω_1 and Ω_δ, provided that Λ satisfies (5.1) and that the boundaries of Ω_1 and Ω_δ are smooth. Moreover, if $\delta > \delta_0 > 0$ then the jump in the coefficients, along $\partial\Omega_\delta$, is finite and the $\|p\|_{H^2(\Omega_1)}$ and $\|p\|_{H^2(\Omega_\delta)}$ norms are likely to be well-behaved. In the present paper we are interested in very small values of δ, i.e. $0 < \delta \ll 1$. More precisely, assume that the interface problem (2.2)-(2.5) is solved by the finite element method. Then we want to prove that the error, measured in proper Sobolev norms, does not blow up as δ tends to zero. From this point of view, the size of the constant δ_0 is of no importance. However, if we could estimate the size of the other constants involved in the proofs of the error estimates presented in this paper, then the size of δ_0 would be of major interest.

6. A quantitative error estimate. In the previous section we proved that the solution p of a problem of the form (3.3) is piecewise smooth, provided that the boundaries of the subdomains Ω_1 and Ω_δ are smooth. It turns out that, given proper conditions on the finite element space V_h, this property of p and Theorem 4.3 are sufficient in order to prove an error estimate of the form (2.7).

In this section we will assume that the finite element space V_h consists of piecewise linear functions defined in terms of a mesh T_h on Ω. Clearly, quantitative error estimates similar to (2.7) can be proved for other types of finite element spaces as well. However, in this paper we will concentrate on the piecewise linear case.

As mentioned above, for general finite element spaces V_h an error estimate of the form (2.7) does not hold, we need two specific assumptions on V_h.

A4. We assume that there exist two constants c_{18} and c_{19}, independent of δ and h, such that for all $v \in H^2(\Omega_\delta)$ and for all $u \in H^2(\Omega_1) \cap \{\psi \in H^1(\Omega_1);\ \psi = 0 \text{ on } \partial\Omega\}$ the following inequalities hold

$$\inf_{w_h \in V_{\Omega_\delta,h}} \|v - w_h\|_{H^1(\Omega_\delta)} \le c_{18}\|v\|_{H^2(\Omega_\delta)}h,$$

$$\inf_{w_h \in V_{\Omega_1,h}} \|u - w_h\|_{H^1(\Omega_1)} \le c_{19}\|u\|_{H^2(\Omega_1)}h.$$

Here, $V_{\Omega_\delta,h}$ and $V_{\Omega_1,h}$ are the finite element spaces defined in (3.5).

A5. For every constant $c \in \mathbb{R}$ the function $\psi(x) = c$ for all $x \in \Omega_\delta$ belongs to $V_{\Omega_\delta,h}$.

As mentioned in Section 3, here h represents the global mesh size for the grid T_h. Conditions **A4** and **A5** are typically satisfied if the grid T_h is constructed such that the interface $\partial\Omega_\delta$ coincides with grid-lines of T_h, cf. e.g. Brenner and Scott [4] or Hackbusch [19]. Recall that we assume that $\partial\Omega_\delta$ is smooth. Hence, $\partial\Omega_\delta$ is likely to be curve-linear and must be represented by so-called isoparametric elements.

Now, from Theorem 4.3 we find that

$$\begin{aligned}\|p - p_h\|^2_{H^1(\Omega)} &\le c^2 \inf_{q_h \in V_h} \|p - q_h\|^2_{H^1(\Omega)} \\ &= c^2 \inf_{q_h \in V_h} \left(\|p - q_h\|^2_{H^1(\Omega_1)} + \|p - q_h\|^2_{H^1(\Omega_\delta)}\right), \end{aligned} \tag{6.1}$$

where c is a constant independent of δ and h. Then, by applying Proposition 5.6 and assumptions **A1**-**A5** we obtain the following theorem.

THEOREM 6.1. *Let p and p_h be the solutions of (3.3) and (3.4), respectively. Suppose f satisfies (2.5) and that assumptions* **A1-A5** *hold. Then there exists a positive constant δ_0 such that for every $\delta \in (0, \delta_0)$ the following error estimate holds*

$$\|p - p_h\|_{H^1(\Omega)} \le ch, \tag{6.2}$$

where c is a constant independent of δ and h.

Results of this flavour are discussed by various authors, cf. e.g. Hackbusch [19, Ch. 10], but for completeness we will present a proof in an appendix. Normally, error estimates of this form are derived by applying an interpolation argument. However, in this paper we prove Theorem 6.1 by utilizing Lemma 4.2, see appendix **A.1.**

7. Numerical experiments. Now we turn our attention to three simple numerical experiments illustrating the theoretical results presented above. Recall assumption (2.5), that the source term f is equal to zero in the low permeable zone Ω_δ. The main purpose of the two first examples, presented in cases I and II below, is to show how the convergence properties of the finite element approximations p_h of p are influenced by this property of f. In the third experiment, Case III, we will consider a problem with smooth coefficients of large variation.

Since Λ_δ has a jump discontinuity at the boundary $\partial\Omega_\delta$ of Ω_δ, cf. equation (2.2) and Figure 2.1, it is difficult to find the analytical solution of problems of the form (2.3)-(2.5). Hence, for the sake of simplicity, we will only consider one-dimensional model problems in this section. These model problems have been discretized using standard piecewise linear finite elements. All computations have been carried out in Matlab on a HP 9000/735 workstation.

7.1. Case I. Let $p = p(x)$ for $x \in [0,3]$ be the weak solution of the following two-point boundary value problem[3]

$$\begin{aligned} &(k(x)p'(x))' = f(x) \quad \text{for } 0 < x < 3, \\ &p(0) = 0 \quad \text{and} \quad p(3) = 1. \end{aligned} \tag{7.1}$$

Here, k and f are given functions defined by

$$(f(x), k(x)) = \begin{cases} (-1,1) & \text{for } 0 \le x \le 1, \\ (0,\delta) & \text{for } 1 < x < 2, \\ (-1,1) & \text{for } 2 \le x \le 3, \end{cases} \tag{7.2}$$

where δ is a positive constant. Clearly, (7.1)-(7.2) is a one-dimensional version of a problem of the form (2.2)-(2.5) with a non-homogeneous Dirichlet boundary condition at $x = 3$. In this case, it is easy to verify that the weak solution of this problem is given by

$$p(x) = \begin{cases} \frac{3\delta+1}{2\delta+1}x - \frac{1}{2}x^2 & \text{for } 0 \le x \le 1, \\ \left(\frac{3\delta+1}{2\delta+1} - \frac{1}{2}\right) + \frac{1}{2\delta+1}(x-1) & \text{for } 1 < x < 2, \\ -\left(\frac{3\delta}{2\delta+1} + \frac{1}{2}\right) + \frac{5\delta+2}{2\delta+1}x - \frac{1}{2}x^2 & \text{for } 2 \le x \le 3. \end{cases} \tag{7.3}$$

Furthermore, Table 7.1 shows the error $\|p - p_h\|_{H^1(\Omega)}$ for various values of δ and h. Clearly, the error is almost independent of δ and of order $O(h)$. This is in agreement with Theorem 4.3 and Theorem 6.1. Moreover, in Figure 7.1 we have plotted the weak solution p of (7.1)-(7.2) for $\delta = 1/2, 1/16$. Clearly, p is well-behaved as δ tends to zero, cf. also (7.3).

7.2. Case II. Next, we consider the two-point boundary value problem (7.1) with coefficient function k defined as in (7.2) and source term f given by

$$f(x) = -1 \quad \text{for all } x \in (0,3).$$

[3] This example was studied from an analytical point of view in [24].

	$\delta = 1/2$	$\delta = 1/4$	$\delta = 1/8$	$\delta = 1/16$
h	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$
10^{-1}	0.0439	0.0435	0.0432	0.0431
20^{-1}	0.0212	0.0211	0.0210	0.0210
40^{-1}	0.0104	0.0104	0.0104	0.0104
80^{-1}	0.0052	0.0051	0.0051	0.0051
160^{-1}	0.0026	0.0026	0.0026	0.0026

TABLE 7.1

The table shows the numerical results for our 1D test problem studied in Case I, i.e. $f(x) = 0$ for all $x \in (1, 2)$.

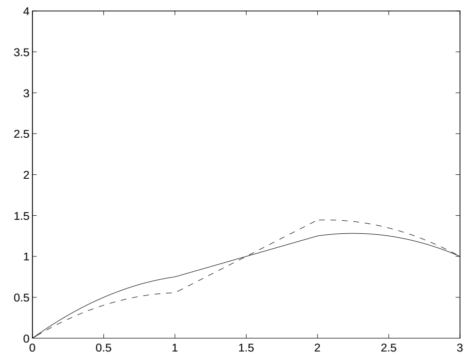

FIG. 7.1. *The figure shows the analytical solution of the problem considered in Case I. The solid and dashed lines are plots of p for $\delta = 1/2$ and $\delta = 1/16$, respectively.*

That is, condition (2.5) for the source term f is violated. The weak solution of this problem is

$$p(x) = \begin{cases} \frac{8\delta+3}{4\delta+2}x - \frac{1}{2}x^2 & \text{for } 0 \le x \le 1, \\ \frac{3\delta^2-2\delta-1}{\delta(2\delta+1)} + \frac{8\delta+3}{\delta(4\delta+2)}x - \frac{1}{2\delta}x^2 & \text{for } 1 < x < 2, \\ \frac{1-\delta}{2\delta+1} + \frac{8\delta+3}{4\delta+2}x - \frac{1}{2}x^2 & \text{for } 2 \le x \le 3, \end{cases} \tag{7.4}$$

and we observe that $p(x) \to \infty$ for $x \in (1, 2)$ if $\delta \to 0$, cf. Figure 7.2.

Table 7.2 shows that the error $\Vert p - p_h \Vert_{H^1(\Omega)}$ increases rapidly as δ decreases. This experiment indicates that a condition like (2.5) is needed in order to obtain error estimates of the form (4.9) and (6.2).

	$\delta = 1/2$	$\delta = 1/4$	$\delta = 1/8$	$\delta = 1/16$
h	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$	$\Vert p - p_h \Vert_{H^1(\Omega)}$
10^{-1}	0.0708	0.1226	0.2347	0.4640
20^{-1}	0.0354	0.0612	0.1173	0.2319
40^{-1}	0.0177	0.0306	0.0586	0.1159
80^{-1}	0.0088	0.0153	0.0293	0.0580
160^{-1}	0.0044	0.0077	0.0147	0.0290

TABLE 7.2

The table shows the numerical results for our 1D test problem studied in Case II, i.e. $f(x) = -1$ for all $x \in (1, 2)$.

7.3. Case III. In the theory developed in sections 4–6, we allow Λ_δ to have large and discontinuous variations. In fact, it is assumed that Λ_δ attains values on certain levels, say $O(1)$ and $O(\delta)$. More precisely, we showed that the error bounds,

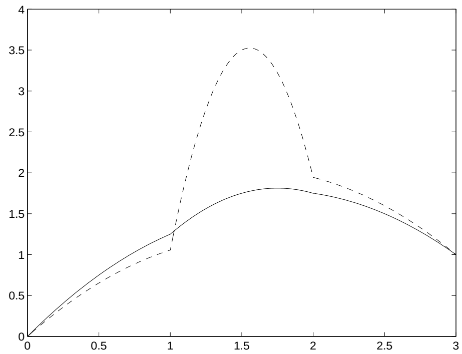

FIG. 7.2. *The figure shows the analytical solution of the problem considered in Case II. The solid and dashed lines are plots of p for $\delta = 1/2$ and $\delta = 1/16$, respectively.*

presented in Theorem 4.3 and Theorem 6.1, hold for mobility functions Λ_δ on the form (2.2). Are similar results valid for smooth coefficient functions?

Again we consider a two-point boundary value problem on the form (7.1), with source term $f(x) = 0$ for all $x \in (0, 3)$. That is, condition (2.5) is satisfied. However, in this case the coefficient function k is given by

$$k(x) = \left(x - \frac{3}{2}\right)^2 + \delta \quad \text{for all } x \in (0,3),$$

and hence k is smooth and of large variation, i.e not on the form (2.2).

It is easy to verify that the analytical solution p of this problem is

$$p(x) = \left(2 \arctan\left(\frac{3/2}{\sqrt{\delta}}\right)\right)^{-1} \arctan\left(\frac{x - 3/2}{\sqrt{\delta}}\right) + \frac{1}{2} \quad \text{for all } x \in (0,3),$$

and we observe that $p'(3/2) \to \infty$ as $\delta \to 0$, see Figure 7.3. Thus, indicating that the error $\|p - p_h\|_{H^1(\Omega)}$ might increase as δ tend towards zero, which is confirmed by Table 7.3. Hence, it seems like error bounds, similar to theorems (4.3) and 6.1, are not obtainable for problems with smooth coefficients of large variation.

	$\delta = 1/2$	$\delta = 1/4$	$\delta = 1/8$	$\delta = 1/16$
h	$\|p - p_h\|_{H^1(\Omega)}$	$\|p - p_h\|_{H^1(\Omega)}$	$\|p - p_h\|_{H^1(\Omega)}$	$\|p - p_h\|_{H^1(\Omega)}$
10^{-1}	0.0188	0.0288	0.0452	0.0722
20^{-1}	0.0094	0.0144	0.0227	0.0363
40^{-1}	0.0047	0.0072	0.0113	0.0182
80^{-1}	0.0023	0.0036	0.0057	0.0091
160^{-1}	0.0012	0.0018	0.0028	0.0045

TABLE 7.3
The table shows the numerical results for our 1D test problem studied in Case III.

Acknowledgement. The author wishes to thank Prof. Tveito for valuable discussions and for encouraging the work presented in this paper. He would also like to thank Prof. Winther and Prof. Langer for numerous comments and suggestions improving this paper.

Appendix A.

A.1. Proof of Theorem 6.1.

We start by constructing the best approximation v_h in $V_{\Omega_\delta,h}$ of $p|_{\Omega_\delta}$, where $V_{\Omega_\delta,h}$ is

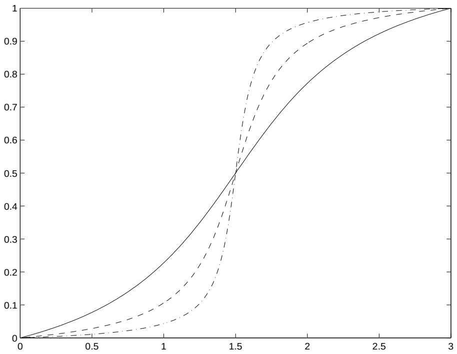

FIG. 7.3. *The figure shows the analytical solution of the problem considered in Case III. The solid, dashed and dashed-dotted lines are plots of p for $\delta = 1/2$, $\delta = 1/16$ and $\delta = 1/100$, respectively.*

the finite element space defined in (3.5). To this end, consider the function space

$$V^0_{\Omega_\delta} = \{\psi \in H^1(\Omega_\delta);\ \int_{\Omega_\delta} \psi\ dx = 0\}.$$

In this space Poincaré's inequality holds, i.e. there exists a constant c_{20} such that

$$\int_{\Omega_\delta} \psi^2\ dx \le c_{20} \int_{\Omega_\delta} |\nabla\psi|^2\ dx \quad \text{for all } \psi \in V^0_{\Omega_\delta}, \tag{A.1}$$

cf. e.g. Dautray and Lions [10]. Next, we introduce the set $V^0_{\Omega_\delta,h}$ of discrete functions in $V^0_{\Omega_\delta}$, i.e.

$$V^0_{\Omega_\delta,h} = \{\psi \in V_{\Omega_\delta,h};\ \int_{\Omega_\delta} \psi\ dx = 0\} \subset V^0_{\Omega_\delta}.$$

From (A.1) and (3.2) it follows that

$$[\varphi,\psi]_{1,\Omega_\delta} = \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\varphi)\ dx \tag{A.2}$$

defines an inner-product on $V^0_{\Omega_\delta}$. The associated norm is

$$\|\psi\|_{1,\Omega_\delta} = \left(\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\psi)\ dx\right)^{1/2}.$$

Thus, the $[\cdot,\cdot]_{1,\Omega_\delta}$-projection r_h of $r = p|_{\Omega_\delta} - |\Omega_\delta|^{-1}\int_{\Omega_\delta} p\ dx \in V^0_{\Omega_\delta}$ in $V^0_{\Omega_\delta,h}$ is well defined and satisfies

$$[r - r_h, \psi]_{1,\Omega_\delta} = 0 \quad \text{for all } \psi \in V^0_{\Omega_\delta,h}.$$

From assumption **A5** we find that $v_h = r_h + |\Omega_\delta|^{-1}\int_{\Omega_\delta} p\ dx$ belongs to $V_{\Omega_\delta,h}$. Furthermore, $p|_{\Omega_\delta} - v_h = r - r_h$, and hence

$$[p - v_h, \psi]_{1,\Omega_\delta} = [r - r_h, \psi]_{1,\Omega_\delta} = 0 \quad \text{for all } \psi \in V^0_{\Omega_\delta,h}. \tag{A.3}$$

Now, notice that the inner-product $[\cdot,\cdot]_{1,\Omega_\delta}$, initially defined on $V^0_{\Omega_\delta}$, also defines a semi-inner-product on $H^1(\Omega_\delta)$ and on $V_{\Omega_\delta,h} \subset H^1(\Omega_\delta)$. Let $\psi \in V_{\Omega_\delta,h}$ be arbitrary, then $\psi - |\Omega_\delta|^{-1}\int_{\Omega_\delta} \psi\ dx \in V^0_{\Omega_\delta,h}$, and from (A.3) and the definition (A.2) of the $[\cdot,\cdot]_{1,\Omega\delta}$ inner-product it follows that

$$[p - v_h, \psi]_{1,\Omega\delta} = [p - v_h, \psi - |\Omega_\delta|^{-1}\int_{\Omega_\delta} \psi\ dx]_{1,\Omega\delta} + [p - v_h, |\Omega_\delta|^{-1}\int_{\Omega_\delta} \psi\ dx]_{1,\Omega\delta} = 0.$$

Consequently,

$$[p - v_h, \psi]_{1,\Omega\delta} = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h}, \tag{A.4}$$

which together with inequality (3.2) implies that

$$\|p - v_h\|_{1,\Omega_\delta} = \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{1,\Omega_\delta} \le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)}. \tag{A.5}$$

That is, v_h is a best approximation, measured in the $\|\cdot\|_{1,\Omega_\delta}$-norm, in $V_{\Omega_\delta,h}$ of $p|_{\Omega_\delta}$. Since $p|_{\Omega_\delta} - v_h = r - r_h \in V^0_{\Omega_\delta}$ we find from inequalities (A.1) and (A.5) that

$$\|p - v_h\|_{H^1(\Omega_\delta)} \le c_{21} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)}, \tag{A.6}$$

where c_{21} is a constant independent of δ and h.

Next, equation (A.4) implies that

$$[v_h, \psi]_{1,\Omega_\delta} = [p, \psi]_{1,\Omega_\delta} \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H^1_0(\Omega_\delta).$$

Hence, by assumption **A2** and Lemma 4.2, part **a)**, we find that

$$\int_{\Omega_\delta} \psi \cdot (\Lambda \nabla v_h)\, dx = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H^1_0(\Omega_\delta),$$

and it follows that v_h is a so-called discrete Λ-harmonic function in Ω_δ. Moreover, from assumptions **A1** and **A2** we conclude that

$$\|v_h - q_h\|_{H^1(\Omega_\delta)} \le c_1 \|v_h - q_h\|_{H^{1/2}(\partial\Omega_\delta)} \quad \text{for all } q_h \in S_{1,h}, \tag{A.7}$$

cf. also Lemma 4.1.

Let $q_h \in S_{1,h}$ be arbitrary. Now, it turns out that we can use inequality (A.7) to bound the size of $\|p - q_h\|_{1,\Omega_\delta}$ on Ω_δ by the size of $\|p - q_h\|_{H^1(\Omega_1)}$ on Ω_1. More precisely, the triangle inequality, (A.5), (3.2), (A.7), the trace theorem and (A.6) implies that

$$\begin{aligned}
\|p - q_h\|_{1,\Omega_\delta} &\le \|p - v_h\|_{1,\Omega_\delta} + \|v_h - q_h\|_{1,\Omega_\delta} \\
&\le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + \sqrt{M}\|v_h - q_h\|_{H^1(\Omega_\delta)} \\
&\le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + c_1\sqrt{M}\|v_h - q_h\|_{H^{1/2}(\partial\Omega_\delta)} \\
&\le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + c_1\sqrt{M}\|p - v_h\|_{H^{1/2}(\partial\Omega_\delta)} \\
&\quad + c_1\sqrt{M}\|p - q_h\|_{H^{1/2}(\partial\Omega_\delta)} \\
&\le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + c_1\|T_{\Omega_\delta}\|\sqrt{M}\|p - v_h\|_{H^1(\Omega_\delta)} \\
&\quad + c_1\|T_{\Omega_1}\|\sqrt{M}\|p - q_h\|_{H^1(\Omega_1)} \\
&\le \sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + c_1 c_{21}\|T_{\Omega_\delta}\|\sqrt{M} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} \\
&\quad + c_1\|T_{\Omega_1}\|\sqrt{M}\|p - q_h\|_{H^1(\Omega_1)} \\
&= c_{22} \inf_{w_h \in V_{\Omega_\delta,h}} \|p - w_h\|_{H^1(\Omega_\delta)} + c_1\|T_{\Omega_1}\|\sqrt{M}\|p - q_h\|_{H^1(\Omega_1)}. \qquad \text{(A.8)}
\end{aligned}$$

Clearly, the constant c_{22} is independent of δ and h. Furthermore, from Proposition 5.6 and assumption **A4** we find that

$$\begin{aligned}
&\left(\int_{\Omega_\delta} \nabla(p - q_h) \cdot (\Lambda\nabla(p - q_h)\, dx\right)^{1/2} \\
&\le c_{23}\|p\|_{H^2(\Omega_\delta)} h + c_1\|T_{\Omega_1}\|\sqrt{M}\|p - q_h\|_{H^1(\Omega_1)} \quad \text{for all } q_h \in S_{1,h}, \qquad \text{(A.9)}
\end{aligned}$$

where $c_{23} = c_{22}c_{18}$.

Recall Lemma 4.2, part **b)**, that the $[\cdot,\cdot]_1$-projection τ_h of p in V_h belongs to $S_{1,h}$. Therefore, it follows from (4.16), (A.9) and (3.2) that

$$\|p - p_h\|_1 \le c_{24}\|p - \tau_h\|_1 \le c_{24} \inf_{q_h \in S_{1,h}} \|p - q_h\|_1$$

$$= c_{24} \inf_{q_h \in S_{1,h}} \left(\int_{\Omega_\delta} \nabla(p - q_h) \cdot (\Lambda\nabla(p - q_h))\, dx + \int_{\Omega_1} \nabla(p - q_h) \cdot (\Lambda\nabla(p - q_h)\, dx \right)^{1/2}$$

$$\le c_{24} \inf_{q_h \in S_{1,h}} \left((c_{23}\|p\|_{H^2(\Omega_\delta)}h + c_{25}\|p - q_h\|_{H^1(\Omega_1)})^2 + M\|p - q_h\|^2_{H^1(\Omega_1)} \right)^{1/2},$$

where c_{24} and c_{25} are constants independent of δ and h. By assumption **A1** every function $w_h \in V_{\Omega_1,h}$ can be extended to a function $q_h \in S_{1,h}$ by constructing the so-called discrete Λ-harmonic extension of w_h. Hence, from assumption **A4** we conclude that

$$\|p - p_h\|_1$$

$$\le c_{24} \inf_{w_h \in V_{\Omega_1,h}} \left((c_{23}\|p\|_{H^2(\Omega_\delta)}h + c_{25}\|p - w_h\|_{H^1(\Omega_1)})^2 + M\|p - w_h\|^2_{H^1(\Omega_1)} \right)^{1/2}$$

$$\le c_{24} \left((c_{12}\|p\|_{H^2(\Omega_\delta)}h + c_{25}c_{19}\|p\|_{H^2(\Omega_1)}h)^2 + M(c_{19}\|p\|_{H^2(\Omega_1)}h)^2 \right)^{1/2}.$$

Recall Proposition 5.6, that the Sobolev norms $\|p\|_{H^2(\Omega_\delta)}$ and $\|p\|_{H^2(\Omega_1)}$ are bounded independently of δ. Hence, we conclude that there exists a constant c_{26}, not depending on δ or h, such that

$$\|p - p_h\|_1 \le c_{26}h.$$

The desired result now follows from inequality (4.6).

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