Norsk Regnesentral
NORWEGIAN COMPUTING CENTER

www.nr.no

When Can Shape and Scale Parameters of a 3D Variogram Be Estimated?

Pål Dahle, Odd Kolbjørnsen, and Petter Abrahamsen

Norwegian Computing Center

Outline of Talk

- Why use parametric variograms?
- How to fit parametric variograms to data.
- A synthetic case study.
- Concluding remarks.

Why Use Parametric Variograms?

- Problem: Empirical variogram estimates are not necessarily negative definite.
- Remedy: Fit a parametric variogram to data.

How to Fit Parametric Variograms

We must choose:

1. Empirical variogram estimator.
2. Parametric variogram to fit.
3. Misfit estimator (error estimator).
4. Optimisation algorithm.

Empirical Variogram Estimator

Methods of moments estimator by Matheron (1963)

$$\hat{\gamma}(\mathbf{h}) = \frac{1}{2N_{\mathbf{h}}} \sum_{N(\mathbf{h})} \left(Z(\mathbf{x}_i) - Z(\mathbf{x}_j)\right)^2, \qquad \mathbf{h} \in \mathbb{R}^3$$

where

$$N(\mathbf{h}) = \{(\mathbf{x}_i, \mathbf{x}_j) : \mathbf{x}_i - \mathbf{x}_j = \mathbf{h}\}$$

and $N_{\mathbf{h}}$ is the cardinality of $N(\mathbf{h})$.

Empirical Variogram Estimator

Robustified estimator by Hawkins and Cressie (1984)

$$\bar{\gamma}(\mathbf{h}) = \frac{1}{2g(N_{\mathbf{h}})} \left\{ \sum_{N(\mathbf{h})} |Z(\mathbf{x}_i) - Z(\mathbf{x}_j)|^{\frac{1}{2}} \right\}^4, \quad \mathbf{h} \in \mathbb{R}^3$$

where

$$g(N_{\mathbf{h}}) = N_{\mathbf{h}}^4 \left(0.457 + 0.494/N_{\mathbf{h}}\right).$$

Norsk Regnesentral
NORWEGIAN COMPUTING CENTER

Parametric Variograms

Spherical:

$$\gamma(h; a, \sigma) = \sigma^2 \left[1.5(h/a) - 0.5(h/a)^3\right]$$

General exponential:

$$\gamma(h; a, \sigma, \alpha) = \sigma^2 \left[1 - \exp\left(-3(h/a)^\alpha\right)\right]$$

where $2 \geq \alpha > 0$

The Misfit Estimator

Approximate weighted least mean square (Cressie 1985):

$$F(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{N_{\mathbf{h}_i}}{\gamma(\mathbf{h}_i;\boldsymbol{\theta})^2} \left[\gamma^*(\mathbf{h}_i) - \gamma(\mathbf{h}_i;\boldsymbol{\theta})\right]^2$$

empirical estimate

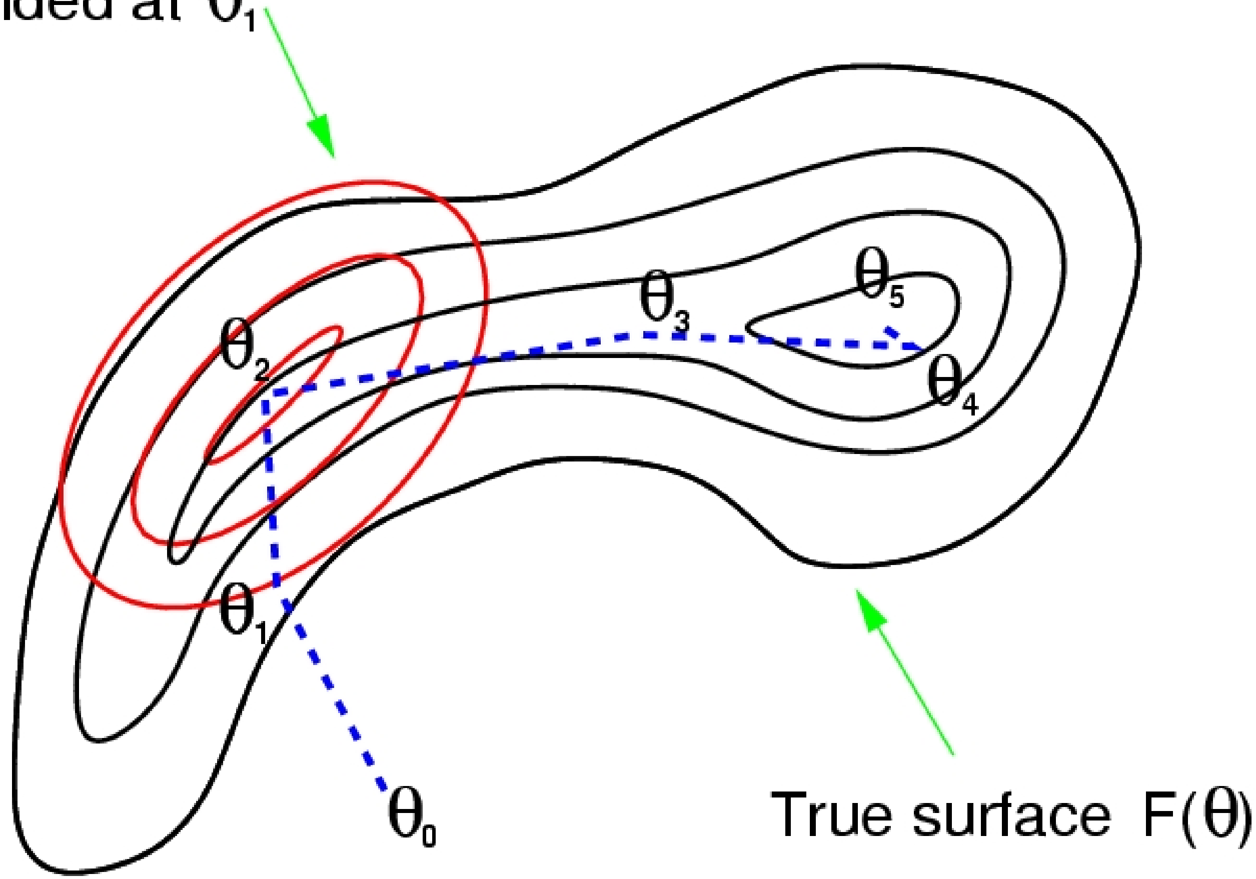
parametric variogram

The lags $\mathbf{h}_i$=1,2,… are the lags for which the empirical estimate and the model are to be compared.

Second-order Optimisation

$$F(\boldsymbol{\theta} + \mathbf{p}) \approx F(\boldsymbol{\theta}) + \mathbf{g}^T\mathbf{p} + \frac{1}{2}\mathbf{p}^T\mathbf{G}\mathbf{p}$$

Approximate surface expanded at θ_1



Case Study: Main Idea

1. Simulate 100 Gaussian random fields with known variogram type.
2. Estimate empirical variogram from “well data”.
3. Fit parametric variograms to empirical variogram.
4. Evaluate the quality using the “Root Mean Square Error”:

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{100}(\hat{\theta}_i - \theta)^2}{100}}$$

parameter estimator

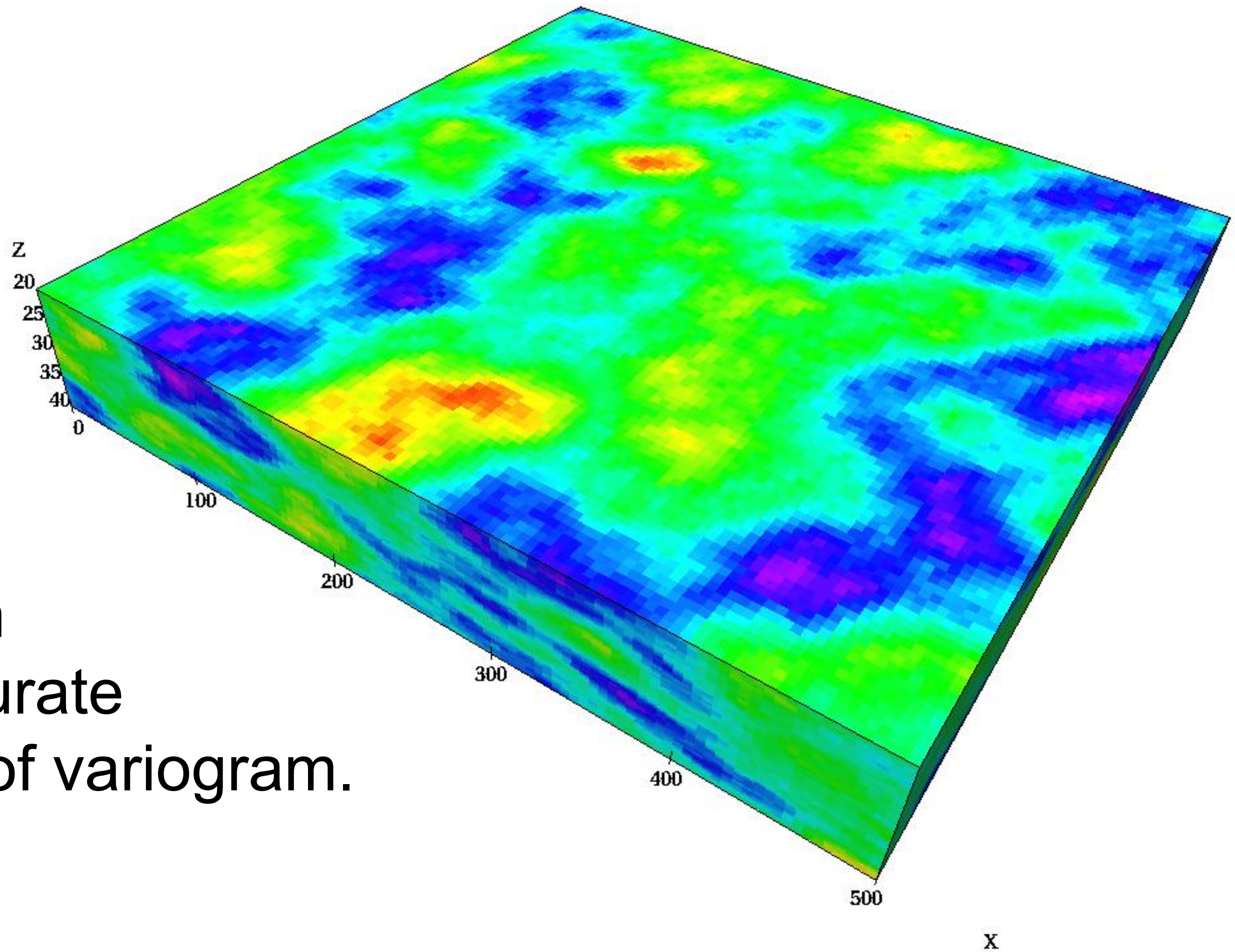
variogram parameter as specified for field

Gaussian Random Field Setup

- 100 fields were generated for each variogram type:
  - Spherical.
  - Exponential.
  - General exponential (α=1.5).
  - Gaussian.
- Grid: (500m,500m,20m) with 100x100x50 cells.
- Variogram parameters:

1. range	**200m**	azimuth	**60º**
2. range	**100m**	dip	**3º**
3. range	**10m**	sill	**1**

Sample Gaussian Random Field



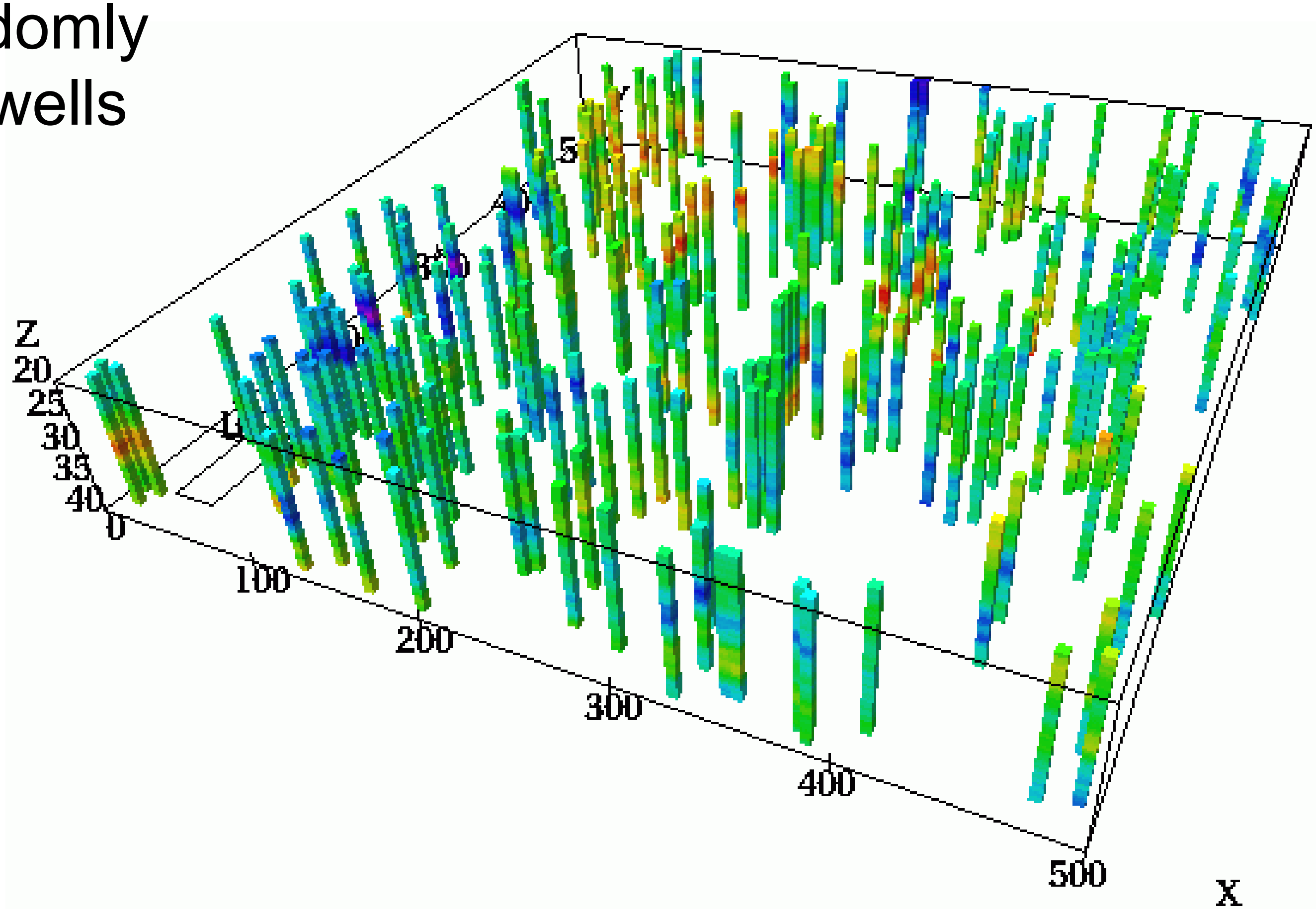
FFT algorithm
with very accurate
reproduction of variogram.

Data Set

- Sets of 10, 50, 100, and 200 randomly located wells.
  - 50 data in each well.
- Gives data sets of 500, 2500, 5000, and 10000 data.

Sample Wells

200 randomly
located wells

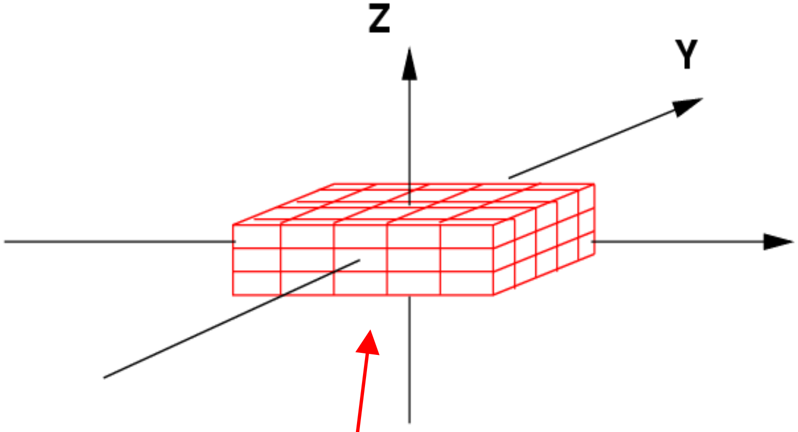


Empirical Variograms

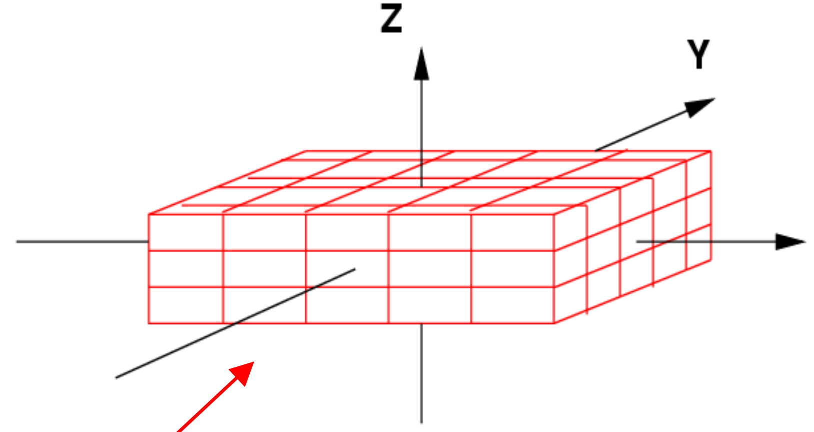
- Use both Matheron and Hawkins/Cressie estimators.
- Lag-grids varying in size and mesh.
- Gives a total of 57,600 empirical variograms:
  - 4 variogram types.
  - 100 repetitions of each field.
  - 4 data sets from each field.
  - 2 estimators.
  - 6*3 different lag-grid layouts.

Empirical Variograms: Lag Grid Size

short lag vectors



short & medium lag vectors

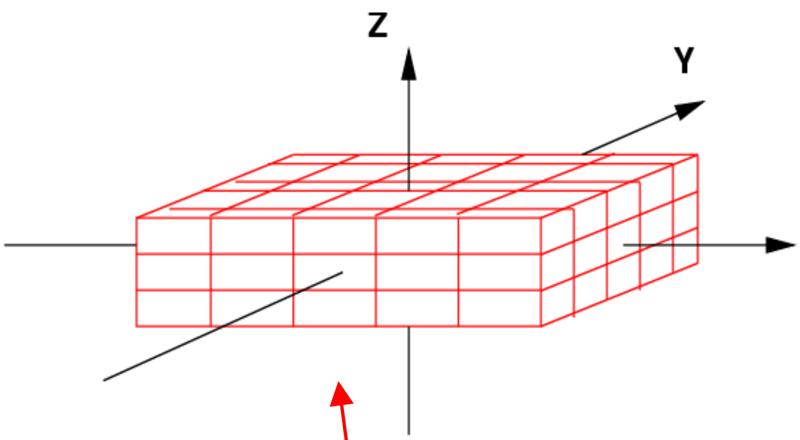


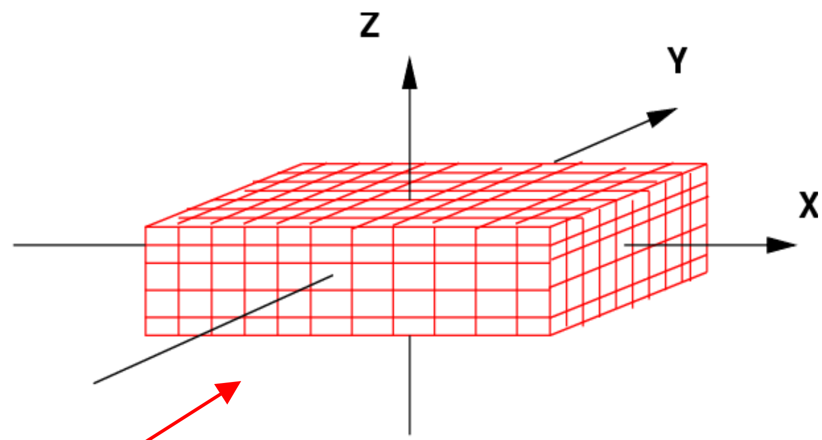
(100m,100m,4m), (300m,300m,10m), (500m,500m,20m)

Empirical Variograms: Lag Grid Mesh

coarse-meshed

not-so-coarse-meshed





(5x5x3), (11x11x5), (21x21x11), (41x41x21), (81x81x41), (161x161x81).

Optimisation Strategies

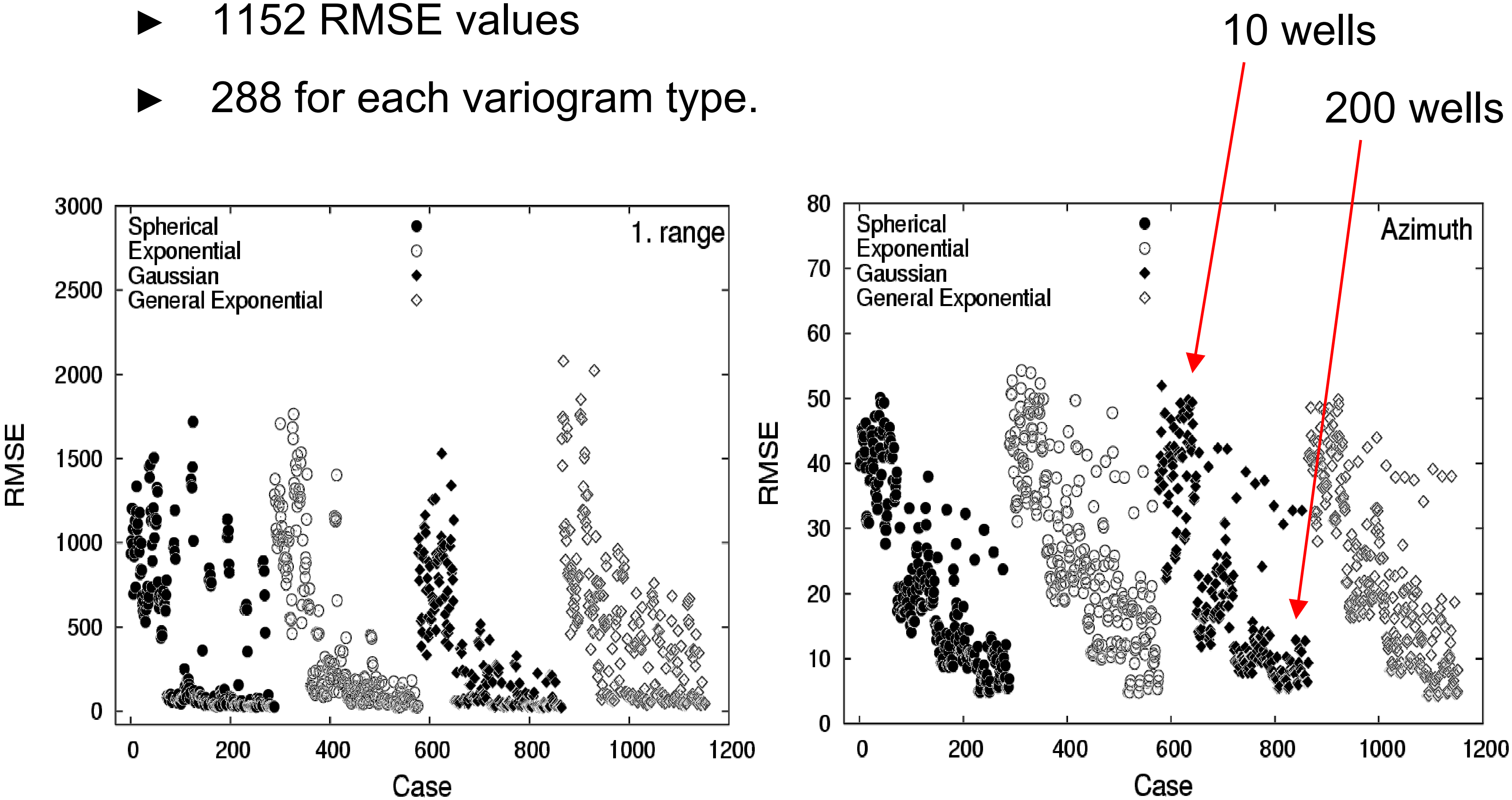
A) Optimise all variogram parameters simultaneously.

B) Optimise all parameters except sill.
   1) Estimate the sill directly from data.
   2) Use this estimate in the parametric variogram when the other parameters are optimised.

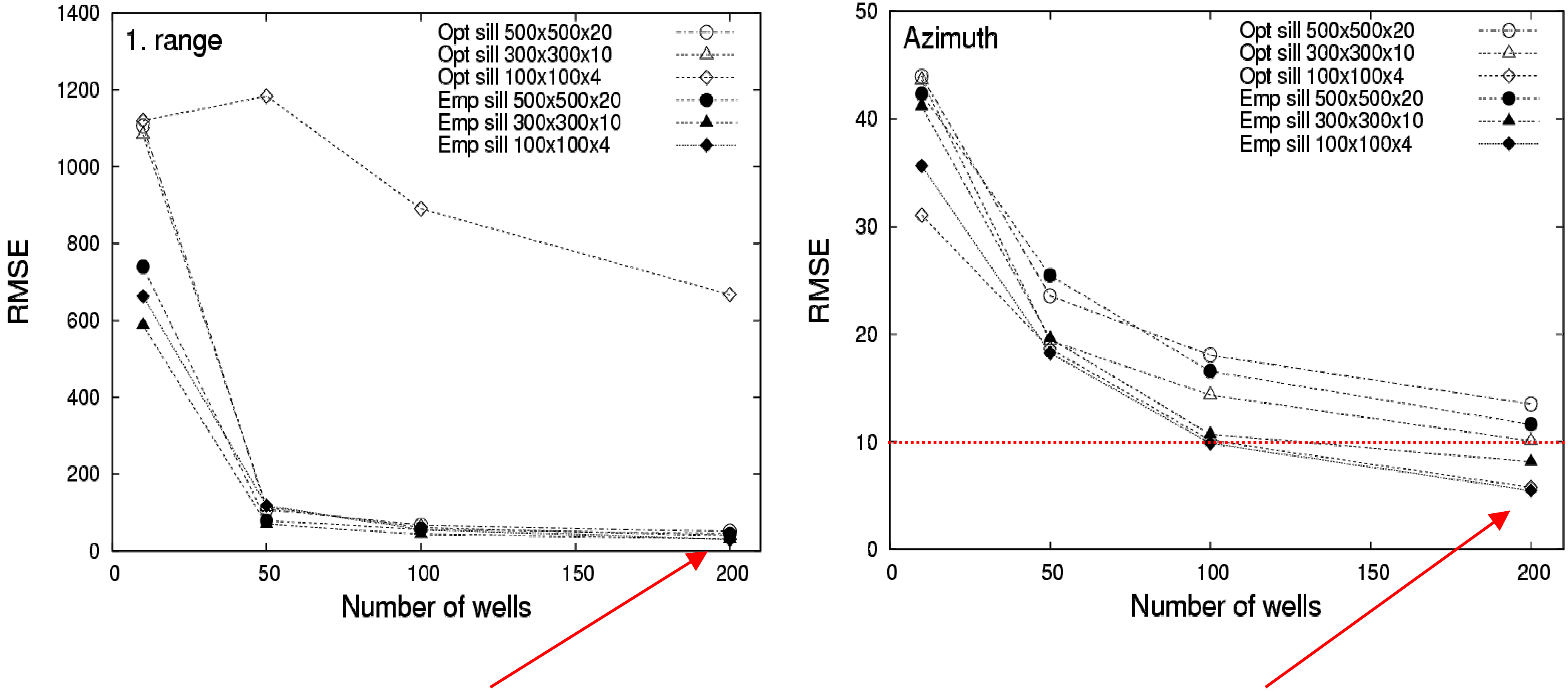
Number of variogram fits: 115,200

RMSE for Azimuth and 1.Range

- 1152 RMSE values
- 288 for each variogram type.



Sensitivity With Respect to Lag-grid Size and Optimisation Strategy



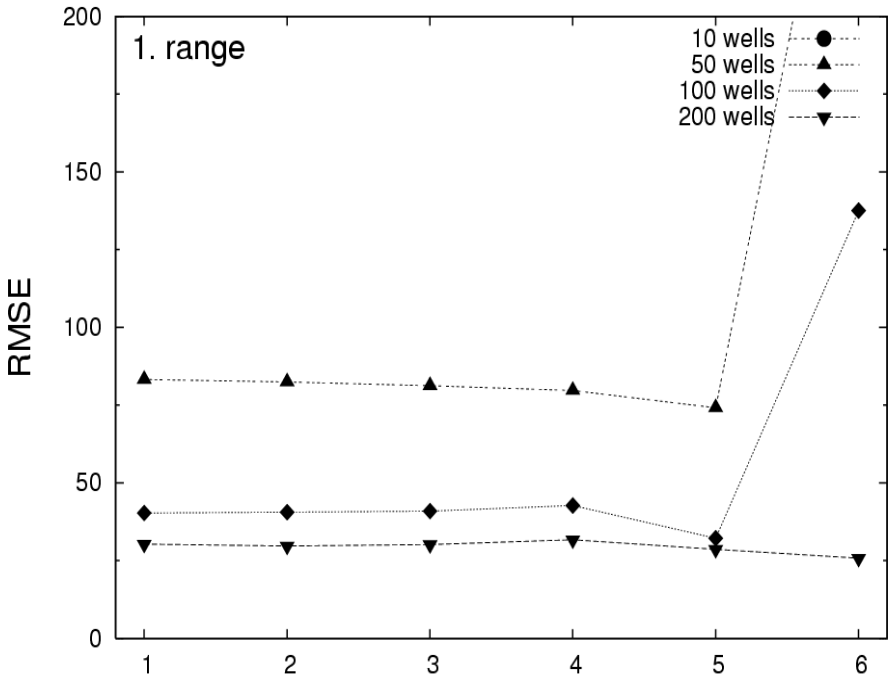
Empirical sill best for ranges

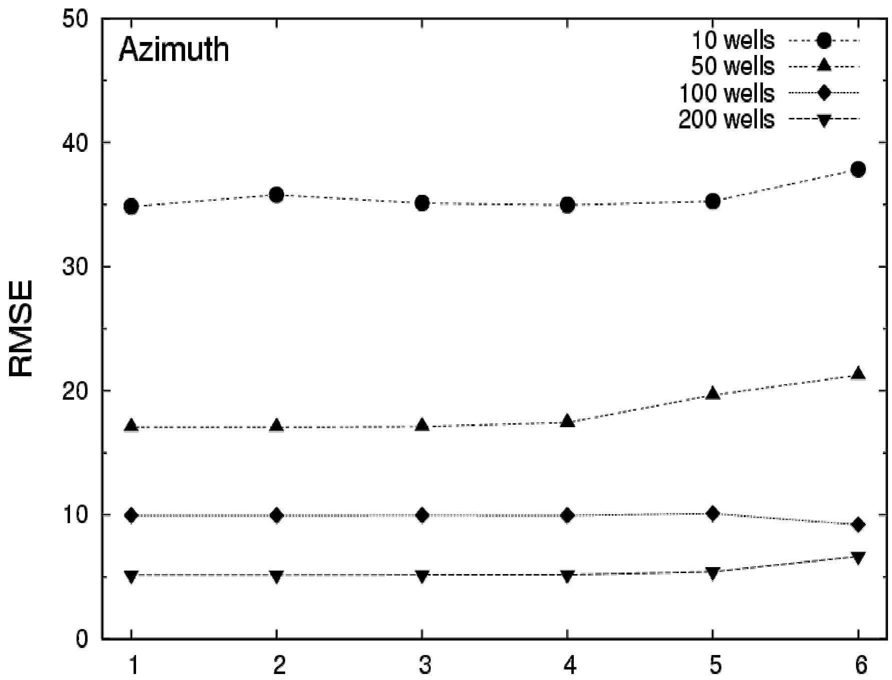
Small lag-grids best for angles

Sensitivity With Respect to Lag-grid Mesh

1 (161x161x81), 2 (81x81x41), 3 (41x41x21), 4 (21x21x11), 5 (11x11x5), 6 (5x5x3).

our choice





coarser lag-grid

coarser lag-grid

RMSE Values With Best Approach

	1. range	2. range	3. range	azimuth	dip	sill	α
10 wells	660	180	2.5	35	6.3	0.26	0.12
50 wells	81	41	1.5	17	1.8	0.15	0.15
100 wells	41	34	1.5	10	1.0	0.14	0.18
200 wells	30	22	1.4	5	0.5	0.13	0.12
orig. value	200	100	10	60	3	1	1.5

Parameter value used for Gaussian random field.

Concluding Remarks

- The sill should be estimated directly from data.
- When estimating empirical variograms:
  - Lag-grid sizes equal to variogram ranges.
  - Lag-grid fairly coarse (41x41x21).
- To get reliable parameter estimates:
  - 200 vertical wells for azimuth and dip.
  - 50 vertical wells for horisontal ranges and sill.
  - 10 vertical wells for vertical range.
- Matheron estimator performs slightly better than Hawkins/Cressie estimator.

Can we identify the variogram type?

		Variogram of Gaussian random field											
		200 wells				50 wells				10 wells			
		Sph	Exp	Gen	Gau	Sph	Exp	Gen	Gau	Sph	Exp	Gen	Gau
Best model	Sph	97	20	–	–	94	64	1	–	69	37	4	–
Best model	Exp	3	80	–	–	5	36	2	–	28	61	1	–
Best model	Gen	–	–	100	–	1	–	96	–	3	1	96	3
Best model	Gau	–	–	–	100	–	–	1	100	–	1	3	97

Matheron or Hawkins/Cressie Estimator?

	1.range		2.range		3.range		azimuth		dip		sill		VarPar	
	bias	rmse	bias	rmse	bias	rmse	bias	rmse	bias	rmse	bias	rmse	bias	rmse
Math	−11.9	29.5	−15.1	19.1	−0.88	1.38	−0.35	4.90	0.16	0.53	−0.03	0.13	0.08	0.10
H/C	−12.5	30.8	−21.3	24.2	−0.93	1.51	−0.07	5.47	0.34	0.65	−0.03	0.13	0.14	0.15