Numerical Valuation of American Options Under the CGMY Process

Ariel Almendral*

Abstract

American put options written on an underlying stock following a Carr-Madan-Geman-Yor (CGMY) process are considered. It is known that American option prices satisfy a Partial Integro-Differential Equation (PIDE) on a moving domain. These equations are reformulated as a Linear Complementarity Problem, and solved iteratively by an implicit-explicit type of iteration based on a convenient splitting of the Integro-Differential operator. The solution to the discrete complementarity problems is found by the Brennan-Schwartz algorithm and computations are accelerated by the Fast Fourier Transform. The method is illustrated throughout a series of numerical experiments.

1 Introduction

In this paper we propose a numerical method to compute American put options, when the underlying asset is modeled by the Carr-Madan-Geman-Yor (CGMY) process considered in [8]. Our contribution is to show experimentally that the implicit-explicit method proposed in [12] for European options may be successfully applied to the computation of American options under Lévy models. A similar splitting was already proposed in [13] for the computation of the American price under the Variance Gamma (VG) process; see also [2].

Matache et al. [17] have previously studied the American pricing problem under the CGMY process. They considered a variational inequality formulation combined with a convenient wavelet basis to compress the stiffness matrix. The approach here is different: we essentially work with a formulation as a Linear Complementarity Problem (LCP), and use standard finite differences. To deal with the singularity of the jump measure at the origin, we first approximate the problem by another problem, where small jumps are substituted by a small Brownian component. Next, we solve the approximated problem iteratively, where for each time step one needs to

*Norwegian Computing Center, Gaustadalleen 23, Postbox 114, Blindern, N-0314, Oslo, Norway (ariel@nr.no).

solve tridiagonal linear complementarity problems. The Fast Fourier Transform (FFT) plays also an important role when computing the convolution integrals fast. The sequence of linear complementarity problems are solved with the help of a simple algorithm proposed by Brennan and Schwartz [6], that works well for the particular case of a put option. We have also verified numerically the recent results in [1] on the smooth-fit principle for general Lévy processes.

A statistical study of financial time series in [8] shows that the diffusion component could in most cases be neglected, provided that the remaining part of the process is of infinite activity and finite variation. We concentrate precisely on the finite variation case, but also allow for a diffusion component, that may be safely omitted without affecting the pricing algorithm.

In Section 2 we briefly introduce the CGMY process, the European and American put option problem, and the related PIDEs. For further information on Lévy processes in finance we refer to the books [11, 20]. An approximation to the equation with a discretization by finite differences is exposed in Section 3 and numerical results are presented in Section 4.

2 The CGMY process as a Lévy process

A Lévy process is a stochastic process with stationary, independent increments. The Lévy-Khintchine theorem (see [19]) provides a characterization of Lévy processes in terms of the characteristic function of the process, namely, there exists a measure ν such that, for all $z \in \mathbb{R}$ and $t \geq 0$, $E(e^{izL_t}) = \exp(t\phi(z))$, where

$$\phi(z) = i\gamma z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}}) d\nu(x). \tag{1}$$

Here $\sigma \geq 0$, $\gamma \in \mathbb{R}$ and ν is a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, x^2) d\nu(x) < \infty$.

Consider a Lévy process $\{L_t\}_{t\geq 0}$ of the form

$$L_t = (r - q + \mu)t + \sigma W_t + Z_t, \tag{2}$$

where r and q are the risk-free interest rate and the continuous dividend paid by the asset, respectively. This process has a drift term controlled by μ, a Brownian component $\{W_t\}_{t\geq 0}$ and a pure-jump component $\{Z_t\}_{t\geq 0}$. In this paper we focus on the case where the Lévy measure in (1) associated to the pure-jump component can be written as $d\nu(x) = k(x)dx$, where the weight $k(x)$ is defined as

$$k(x) = \begin{cases} C\dfrac{\exp(-G|x|)}{|x|^{1+Y}} & \text{if} \quad x < 0, \\[2ex] C\dfrac{\exp(-M|x|)}{|x|^{1+Y}} & \text{if} \quad x > 0, \end{cases} \tag{3}$$

for constants $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. The process $\{Z_t\}_{t\geq 0}$ is known in the literature as the CGMY process [8]; it generalizes a jump-diffusion model by Kou [15] ($Y = -1$) and the VG process [10] ($Y = 0$). The CGMY process is in turn a particular case of the Kobol process studied in [5] and [7], where the constant C is allowed to take on different values on the positive and negative semiaxes.

The characteristic function of the CGMY process may be computed explicitly, see [5, 8]. In this paper, we consider *only* those processes having infinite activity and finite variation, excluding the VG process, that is, $0 < Y < 1$. In such situation one has

$$\phi(z) = (r - q + \mu)iz - \frac{\sigma^2}{2}z^2 + C\Gamma(-Y)\left\{(M - iz)^Y - M^Y + (G + iz)^Y - G^Y\right\}. \quad (4)$$

A market model

Let a market consist of one risky asset $\{S_t\}_{t\geq 0}$ and one bank account $\{B_t\}_{t\geq 0}$. Let us assume that the asset process $\{S_t\}_{t\geq 0}$ evolves according to the geometric law

$$S_t = S_0 \exp(L_t), \quad (5)$$

where $\{L_t\}_{t\geq 0}$ is the Lévy process defined in (2), and the bank account follows the law $B_t = \exp(rt)$. Assume next the existence of some Equivalent Martingale Measure Q (a measure with the same null sets as the market probability, for which the discounted process $\{e^{-(r-q)t}S_t\}_{t\geq 0}$ are martingales). In this paper one works only with a risk-neutral measure Q, where the drift of the Lévy process has been changed. The EMM-condition $E_Q[S_t] = S_0 e^{t(r-q)}$ implies $\phi(-i) = r - q$, so we get the following risk-neutral form for μ:

$$\omega := -\frac{\sigma^2}{2} - C\Gamma(-Y)\left\{(M-1)^Y - M^Y + (G+1)^Y - G^Y\right\}. \quad (6)$$

We keep the same notation for the risk-neutral parameters G and M. The other parameters σ, C and Y are the same in the risk-neutral world, see e.g., [11, 18]. Note that M must be larger than one for ω to be well defined.

2.1 Options in a Lévy market

European vanilla options

Consider a European put option on the asset $\{S_t\}_{t\geq 0}$, with time to expiration T, and strike price K. Let us define the price of a European put option by the formula:

$$v(\tau, s) = e^{-r\tau}E_Q\left[(K - sH_\tau)^+\right], \quad 0 \leq s < \infty, \quad 0 \leq \tau \leq T, \quad (7)$$

where the process $\{H_\tau\}_{\tau\geq 0}$ is the underlying risk-neutral process starting at 1, given by

$$H_\tau := \exp\left[(r-q+\omega)\tau + \sigma W_\tau + Z_\tau\right]. \tag{8}$$

Note that τ means time to expiration $T-t$.

We will not work directly with the asset price s, but rather with its logarithm. Thus, let $x = \ln s$, and define the new function

$$u(\tau, x) := v(\tau, e^x). \tag{9}$$

From a generalization of Ito's formula follows that u satisfies the following Cauchy problem:

$$\begin{cases} u_\tau - \mathcal{L}u = 0, \quad \tau \in (0,T], \quad x \in \mathbb{R}, \\ u(0,x) = (K-e^x)^+, \quad x \in \mathbb{R}, \end{cases} \tag{10}$$

where $\mathcal{L}$ is an integro-differential operator of the form

$$\begin{aligned} \mathcal{L}\varphi := \frac{\sigma^2}{2}\varphi_{xx} + (r-q-\frac{\sigma^2}{2})\varphi_x - r\varphi \\ + \int_{\mathbb{R}} \left[\varphi(\tau, x+y) - \varphi(\tau,x) - (e^y-1)\varphi_x(\tau,x)\right] k(y)dy. \end{aligned} \tag{11}$$

For a derivation of (10) see [5, 18].

American vanilla options

Consider an American put option written on the underlying asset $\{S_t\}_{t\geq 0}$. The price may be found by solving an optimal stopping problem of the form:

$$v(\tau, s) = \sup_{\tau' \in \mathcal{S}_{0,\tau}} E_Q\left[e^{-r\tau'}(K - sH_{\tau'})^+\right]. \tag{12}$$

Here $\mathcal{S}_{0,\tau}$ denotes the set of stopping times taking values in $[0,\tau]$ and $\{H_\tau\}_{\tau\geq 0}$ is the process in (8). The corresponding function u (cf. (9)) satisfies the free-boundary value problem [5, 17]:

$$\begin{cases} u_\tau - \mathcal{L}u = 0, \quad \tau > 0, \quad x > \widetilde{c}(\tau), \\ u(\tau,x) = K - e^x, \quad \tau > 0, \quad x \leq \widetilde{c}(\tau), \\ u(\tau,x) \geq (K-e^x)^+, \quad \tau > 0, \quad x \in \mathbb{R}, \\ u_\tau - \mathcal{L}u \geq 0, \quad \tau > 0, \quad x \in \mathbb{R}, \\ u(0,x) = (K-e^x)^+, \quad x \in \mathbb{R}, \end{cases} \tag{13}$$

where the operator $\mathcal{L}$ is defined in (11) and the free-boundary is given by

$$\widetilde{c}(\tau) = \inf\left\{x \in \mathbb{R} \mid u(\tau, x) > (K - e^x)^+\right\}, \quad \tau \in (0, T]. \tag{14}$$

The set $\{x \in \mathbb{R} \mid x \leq \widetilde{c}(\tau)\}$ is the exercise region for the logarithmic prices. Hence, for asset prices $s \leq \exp(\widetilde{c}(\tau))$, the American put should be exercised.

3 Numerical valuation of the American CGMY price

The function $\widetilde{c}(\tau)$ is not known a-priori, and need to be found as part of the solution. Thus, rather than solving (13) directly, it is more convenient to use another formulation as a so-called Linear Complementarity Problem:

$$\begin{cases} u_\tau - \mathcal{L}u \geq 0 & \text{in } (0, T) \times \mathbb{R}, \\ u \geq \psi & \text{in } [0, T] \times \mathbb{R}, \\ (u_\tau - \mathcal{L}u)\,(u - \psi) = 0 & \text{in } (0, T) \times \mathbb{R}, \\ u(0, x) = \psi(x), & \end{cases} \tag{15}$$

where the initial condition is given by

$$\psi(x) := (K - e^x)^+. \tag{16}$$

Note that the dependency on the free-boundary $\widetilde{c}(\tau)$ has disappeared, but instead we are left with a set of inequalities. The discretization and numerical solution of (15) is from now our main goal. The free-boundary is obtained after computing the solution, by making use of (14).

3.1 Discretization and solution algorithm

The main idea of the method is to approximate the operator (11) by truncating the integral term close to zero and infinity. The truncation around infinity is harmless, as long as a sufficiently large interval is chosen and the price is substituted by the option's intrinsic value outside the computational domain. However, the truncation around zero gives rise to an artificial diffusion that must be taken into account. More precisely, the operator $\mathcal{L}$ may be splitted into the sum of two operators: the first one containing the Black and Scholes operator and the second accounting for the jumps, namely, $\mathcal{L} = \mathcal{L}_{BS} + \mathcal{L}_J$. The jump integral part is in turn splitted into the sum of one operator $\mathcal{P}^\epsilon$ for the integration variable in a neighborhood of the origin, and $\mathcal{Q}^\epsilon$ for the complementary domain. For $\mathcal{P}^\epsilon$ we use Taylor's expansion to write the following approximation:

$$\begin{aligned}(\mathcal{P}^\epsilon\varphi)(\tau,x) &:= \int_{|y|\leq\epsilon}\left[\varphi(\tau,x+y)-\varphi(\tau,x)-(e^y-1)\varphi_x(\tau,x)\right]k(y)dy\\ &= \int_{|y|\leq\epsilon}\left[\varphi(\tau,x+y)-\varphi(\tau,x)-y\varphi_x(\tau,x)-(e^y-1-y)\varphi_x(\tau,x)\right]k(y)dy\\ &\approx (\widetilde{\mathcal{P}}^\epsilon\varphi)(\tau,x) := \frac{\sigma^2(\epsilon)}{2}\varphi_{xx}(\tau,x)-\frac{\sigma^2(\epsilon)}{2}\varphi_x(\tau,x),\end{aligned}$$

with the notation:

$$\sigma^2(\epsilon)=\int_{|y|\leq\epsilon} y^2k(y)dy. \tag{17}$$

That is, $\mathcal{P}^\epsilon$ has been approximated by a convection-diffusion operator $\widetilde{\mathcal{P}}^\epsilon$, with a small diffusion coefficient $\sigma^2(\epsilon)$.

The operator $\mathcal{Q}^\epsilon$ is simply splitted into a sum, given that this operation is now allowed away from the origin:

$$\begin{aligned}(\mathcal{Q}^\epsilon\varphi)(\tau,x) &:= \int_{|y|\geq\epsilon}\left[\varphi(\tau,x+y)-\varphi(\tau,x)-(e^y-1)\varphi_x(\tau,x)\right]k(y)dy\\ &= (\mathcal{J}^\epsilon\varphi)(\tau,x)-\lambda(\epsilon)\varphi(\tau,x)+\omega(\epsilon)\varphi_x(\tau,x),\end{aligned} \tag{18}$$

where we have written $\mathcal{J}^\epsilon$ for the convolution term, and

$$\lambda(\epsilon) = \int_{|y|\geq\epsilon} k(y)dy, \tag{19}$$

$$\omega(\epsilon) = \int_{|y|\geq\epsilon} (1-e^y)k(y)dy. \tag{20}$$

Remark 3.1. These operations have a probabilistic meaning: the pure-jump process has been approximated by a compound Poisson process plus a small Brownian component. As proved in [4], this approximation is valid if and only if $\sigma(\epsilon)/\epsilon \to \infty$, as $\epsilon \to 0$. Note that this condition implies $0 < Y < 1$, excluding therefore the VG process and processes with infinite activity.

An approximation result in [12] states the following. Let $\mathcal{L}^\epsilon := \mathcal{L}_{BS} + \widetilde{\mathcal{P}}^\epsilon + \mathcal{Q}^\epsilon$ and u^ϵ be the solution of the Cauchy problem

$$\begin{cases} u^\epsilon_\tau - \mathcal{L}^\epsilon u^\epsilon = 0,\\ u(0,x)=\psi(x),\end{cases} \tag{21}$$

then there exists a constant $C>0$ such that $|u(\tau,x)-u^\epsilon(\tau,x)| < C\epsilon$, for all τ and x. We use here -without proof- the same approximation to numerically solve an American put option. An indication that this approximation works also for American options is shown in Figure 1, where one observes that the exercise boundary tends to the theoretical perpetual exercise price, when the time to expiration τ is taken large. The proof of this fact is thus an open problem.

Let us focus now on problem (15), but with $\mathcal{L}^\epsilon$ instead of $\mathcal{L}$. One possible idea to discretize this new problem is to apply Euler's scheme in time combined with an implicit-explicit iteration in space. Let the time interval $[0, T]$ be divided into L equal parts, i.e., $\tau_j = j\Delta\tau$ $(j = 0, 1, \ldots, L)$ with $\Delta\tau = T/L$ and define the functions $u^j \approx u(\tau_j, x)$. Let operator $\mathcal{L}^\epsilon$ be splitted as $\mathcal{L}^\epsilon = \mathcal{A} + \mathcal{B}$. We consider the following sequence of problems:

$$\begin{cases} \dfrac{u^{j+1}}{\Delta\tau} - \mathcal{A}u^{j+1} \geq d^j := \dfrac{u^j}{\Delta\tau} + \mathcal{B}u^j, \\ u^{j+1} \geq \psi, \\ \left(\dfrac{u^{j+1}}{\Delta\tau} - \mathcal{A}u^{j+1} - d^j\right)(u^{j+1} - \psi) = 0, \\ u^0 = \psi. \end{cases} \tag{22}$$

That is, given the function u^j, we compute u^{j+1} by solving this integro-differential inequalities. A natural choice for the splitting of $\mathcal{L}^\epsilon$ is the following:

$$\mathcal{A}\varphi := \frac{\sigma^2 + \sigma^2(\epsilon)}{2}\varphi_{xx} + \left[r - q - \frac{\sigma^2 + \sigma^2(\epsilon)}{2} + \omega(\epsilon)\right]\varphi_x - r\varphi \tag{23}$$

$$\mathcal{B}\varphi := \mathcal{J}^\epsilon\varphi - \lambda(\epsilon)\varphi. \tag{24}$$

Observe that the integral term is treated explicitly, whereas the differential part is treated implicitly. This method imposes a stability restriction on the time step; see [12] for a discussion of this issue for the European case.

Spatial discretization of $\mathcal{A}$

Consider a computational domain of the form $[0, T] \times [x_{min}, x_{max}]$. Let $\ln K \in [x_{min}, x_{max}]$ and define the uniform spatial grid $x_i = x_{min} + ih$ $(i = 0, \ldots, N)$ where $h = (x_{max} - x_{min})/N$. Once we have defined the grid, we can discretize $\mathcal{A}$ by standard second order schemes. For the first and second derivatives, the central scheme and the standard 3-point scheme are chosen, respectively. Namely, after introducing the notation $\delta_1(\varphi) := [\varphi_{i+1} - \varphi_{i-1}]/2h$ and $\delta_2(\varphi) := [\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}]/h^2$, where $\varphi_i := \varphi(x_i)$ $(i = 0, 1, \ldots, N)$, we may write

$$(\mathcal{A}\varphi)_i = \beta\delta_2(\varphi) + \gamma\delta_1(\varphi) - r\varphi_i, \tag{25}$$

with the quantities β and γ defined as

$$\beta := \frac{\sigma^2 + \sigma^2(\epsilon)}{2}, \tag{26}$$

$$\gamma := r - q - \frac{\sigma^2 + \sigma^2(\epsilon)}{2} + \omega(\epsilon). \tag{27}$$

We obtain the following coefficients for the implicit part

$$a = -\frac{\beta}{h^2} + \frac{\gamma}{2h}, \tag{28}$$

$$b = \frac{1}{\Delta\tau} + r + \frac{2\beta}{h^2}, \tag{29}$$

$$c = -\frac{\beta}{h^2} - \frac{\gamma}{2h}. \tag{30}$$

The tridiagonal matrix T associated to the implicit part has constant diagonals: b is on the main diagonal, a is on the subdiagonal and c is on the superdiagonal.

From now on, the parameter ϵ is taken as the mesh-size h. The artificial diffusion $\sigma^2(h)$ (cf. (17)) may be approximated by the composite trapezoidal rule on the intervals $[-h, 0]$ and $[0, h]$. This gives

$$\sigma^2(h) \approx \frac{[k(h) + k(-h)]\, h^3}{2}. \tag{31}$$

The quantities $\lambda(h)$ and $\omega(h)$ are approximated in the next paragraph.

Spatial discretization of $\mathcal{B}$

The discretization of $\mathcal{B}$ involves the discretization of $\mathcal{J}^\epsilon$, since $\mathcal{B}\varphi = \mathcal{J}^\epsilon\varphi - \lambda(\epsilon)\varphi$. The discretization of $\mathcal{J}^\epsilon$ is explained in detail in [2]. Briefly, the idea is to truncate the integral to a finite domain and then apply the composite trapezoidal rule, i.e,

$$\begin{aligned} J_i := (\mathcal{J}^\epsilon\varphi)_i &= \int\limits_{|y|\geq h} \varphi(x_i + y)k(y)dy \\ &\approx \int\limits_{h\leq|y|\leq Mh} \varphi(x_i + y)k(y)dy \\ &\approx h \sum_{m=-M}^{M} \varphi_{i+m} k_m \rho_m, \quad i = 0, 1, \ldots, N, \end{aligned} \tag{32}$$

where $k_m = k(mh)$ for $m \neq 0$ and we let $k_0 = 0$. The coefficients obtained from applying the trapezoidal rule are:

$$\rho_m = \begin{cases} 1/2 & \text{if } m \in \{-M, -1, 1, M\}, \\ 1 & \text{otherwise.} \end{cases}$$

It is important to substitute φ by the payoff function ψ outside the computational domain. The computation of the numbers J_i constitutes the main burden of the method, but thanks to the FFT algorithm, this may be done efficiently, see next section. However, N must be an even number, and $M = N/2$, to be able to express this convolution in matrix-vector notation.

Finally, we may use the composite trapezoidal rule to compute an approximation to the numbers $\lambda(h)$ and $\omega(h)$ by simply taking φ in (32) as 1 and $e^y - 1$, respectively.

Fast convolution by FFT

The Fast Fourier Transform is an algorithm that evaluates the Discrete Fourier Transform (DFT) of a vector $f = [f_0, f_2 \ldots, f_{R-1}]$ in $O(R \log R)$ operations.

The Discrete Fourier Transform is defined as:

$$F_k = \sum_{n=0}^{R-1} f_n e^{-i2\pi nk/R}, \quad k = 0, 1, \ldots, R. \tag{33}$$

One of the multiple applications of the DFT is in computing convolutions. Let us first introduce the concept of circulant convolution. Let $\{x_m\}$ and $\{y_m\}$ be two sequences with period R. The convolution sequence $z := x * y$ is defined component-wise as

$$z_n = \sum_{m=0}^{R-1} x_{m-n} y_m. \tag{34}$$

We use now FFT to compute the vector $[z_0, \ldots, z_{R-1}]$. The periodic structure of x allows the derivation of the following simple relation:

$$Z_k = X_k \cdot Y_k, \tag{35}$$

where X, Y and Z denote the Discrete Fourier Transform of the sequences x, y and z respectively. That is, DFT applied to the convolution sequence is equal to the product of the transforms of the original two sequences. The vector $[z_0, \ldots, z_{R-1}]$ may be recovered by means of the Inverse Discrete Fourier Transform (IDFT):

$$z_n = \frac{1}{R} \sum_{k=0}^{R-1} Z_k e^{i2\pi kn/R}, \quad n = 0, 1, \ldots, R. \tag{36}$$

In the language of matrices, a circulant convolution may be seen as the product of a circulant matrix times a vector. For example, let $R = 3$, and use the periodicity $x_k = x_{k+R}$ to write (34) as

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}. \tag{37}$$

A circulant matrix is thus a matrix in which each row is a "circular" shift of the previous row.

We are interested in the convolution (32), where the vector φ is not periodic. The associated matrix is a so-called Toeplitz matrix, which by definition is a matrix that is constant along diagonals. A circulant matrix is hence a particular type of Toeplitz matrix. The next idea is to embed a Toeplitz matrix into a circulant matrix. As an example, let $M = 1$ and $N = 2$, so that the matrix-vector notation for (32) reads

$$\begin{bmatrix} \varphi_1 & \varphi_0 & \varphi_{-1} \\ \varphi_2 & \varphi_1 & \varphi_0 \\ \varphi_3 & \varphi_2 & \varphi_1 \end{bmatrix} \begin{bmatrix} k_1/2 \\ k_0 \\ k_{-1}/2 \end{bmatrix}. \tag{38}$$

The matrix above may be embedded in a circulant matrix C of size 5 in the following way (For computational efficiency of the FFT algorithm, it is advisable to use a circulant matrix whose size is a power of 2.):

$$C = \left[\begin{array}{ccc|cc} \varphi_1 & \varphi_0 & \varphi_{-1} & \varphi_3 & \varphi_2 \\ \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} & \varphi_3 \\ \varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} \\ \hline \varphi_{-1} & \varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 \\ \varphi_0 & \varphi_{-1} & \varphi_3 & \varphi_2 & \varphi_1 \end{array}\right]. \tag{39}$$

If we define the vector $\eta := [k_1/2, k_0, k_{-1}/2, 0, 0]^T$, then the product (38) is the vector consisting of the first three elements in the product $C\eta$. As explained before, a product of a circulant matrix and a vector may be efficiently done by applying the FFT algorithm.

As a summary, following the ideas explained above, it is possible to compute the convolution (32), with $M = N/2$, by "embedding" the resulting matrix into a circulant matrix. The product of a circulant matrix and a vector is carried out in three FFT operations, namely, two DFT and one IDFT.

In paper [3] we applied the FFT algorithm in the computation of European options for Merton's model and Kou's model, and in [2] to find the American price under the Variance Gamma process. For further details on the computation of convolutions by FFT we refer to [21].

Boundary conditions

We used points on the boundary when discretizing the differential operator $\mathcal{A}$. This means that the vector d^j needs to be updated. For a put option, this is done by updating the first and the last entries of d^j as follows:

$$d_1^j \leftarrow d_1^j - a(K - e^{x_{min}}), \quad d_{N-1}^j \leftarrow 0. \tag{40}$$

Discrete LCP

We are now in position to write the discrete inequalities that correspond to the discretization of (22):

$$\begin{cases} Tu^{j+1} \geq d^j, \\ u^{j+1} \geq \psi, \\ (Tu^{j+1} - d^j, u^{j+1} - \psi) = 0, \\ u^0 = \psi, \end{cases} \tag{41}$$

for $j = 0, 1, \ldots, L-1$. The matrix T has entries given by (28)-(30), $d_i^j = u_i^j/\Delta\tau + (\mathcal{J}^\epsilon u^j)_i - \lambda(\epsilon) u_i^j$ $(i = 1, \ldots, N-1)$ with the update (40) and ψ is the vector $[\psi_1, \psi_2, \ldots, \psi_{N-1}]^T$, with $\psi_i = \psi(x_i)$ (cf. (16)). The same letter ψ is used to simplify the notation.

We proceed to explain a simple algorithm to solve (41).

Brennan-Schwartz algorithm for a put option

Let a tridiagonal matrix

$$T = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \tag{42}$$

and vectors $d = [d_1, \ldots, d_n]^T$ and $\psi = [\psi_1, \ldots, \psi_n]^T$ be given. Consider the following problem: Find a vector u satisfying the system

$$\begin{cases} Tu \geq d, \\ u \geq \psi, \\ (Tu - d, u - \psi) = 0. \end{cases} \tag{43}$$

The following algorithm to find u in (43) was proposed by Brennan and Schwartz [6] (for put options) and discussed in detail by Jaillet et al. [14]:

- Step 1: Compute recursively a vector $\widetilde{b}$ as

$$\begin{aligned} \widetilde{b}_n &= b_n, \\ \widetilde{b}_{j-1} &= b_{j-1} - c_{j-1} a_j / \widetilde{b}_j, \quad j = n, \ldots, 2. \end{aligned}$$

- Step 2: Compute recursively a vector $\widetilde{d}$ as

$$\begin{aligned} \widetilde{d}_n &= d_n, \\ \widetilde{d}_{j-1} &= d_{j-1} - c_{j-1} \widetilde{d}_j / \widetilde{b}_j, \quad j = n, \ldots, 2. \end{aligned}$$

- Step 3: Compute u forward as follows:

$$\begin{aligned} u_1 &= \max\left[\widetilde{d}_1/b_1 \;,\; \psi_1\right], \\ u_j &= \max\left[\left(\widetilde{d}_j - a_j u_{j-1}\right)/\widetilde{b}_j \;,\; \psi_j\right], \quad j = 2, \dots, n. \end{aligned}$$

We apply these three steps with $a_i = a$, $b_i = b$ and $c_i = c$, with a, b, c as in (28)-(30). The splitting proposed in (23)-(24) does not in general guarantee the validity of Brennan-Schwartz algorithm. However, the convection term may be moved to the explicit part of the splitting, so that the conditions of Brennan-Schwartz algorithm hold [3]. The solutions obtained in both ways are the same, to within the discretization error.

4 Numerical experiments

In this section, European and American option prices are computed numerically. In the first experiment we compute an European option (problem (21)) and compare it with the solution obtained by the Carr-Madan formula in [9]; see also the appendix, formula (9). Both solutions are compared in the ℓ_∞-norm, and the results are shown in Table 1. A linear convergence rate is observed, and note that the algorithm computes the European price with an error of one cent in about one second.

N	L	ℓ_∞-error	CPU-time
50	5	0.2675	0.22 s.
100	10	0.1281	0.31 s.
200	20	0.0459	0.34 s.
400	40	0.0160	1.06 s.

Table 1: Linear convergence to exact solution in ℓ_∞-norm and CPU times on a Pentium IV, 1.7Ghz. The parameters are: $r = 0$, $q = 0$, $K = 10$, $T = 1$, $C = 1$, $G = 7$, $M = 9$ and $Y = 0.7$.

A second experiments concerns the verification of the theoretical perpetual exercise price against the asymptotic behavior of the free boundary for some large time to expiry. The asymptotic value s^* of the American put was verified with the aid of a formula in [5], Theorem 3.2 and Theorem 5.1:

$$s^* = \exp(x^*) = K \exp\left\{-\frac{1}{2\pi}\int_{-\infty+i\rho}^{\infty+i\rho} \frac{\ln\left[r + q + \phi_0(z)\right]}{z^2 + iz} dz\right\}, \tag{44}$$

with ρ a positive number (not arbitrary, see [5]) and $\phi_0(z)$ is given by (47). Figure 1 shows two examples of exercise boundaries and their corresponding theoretical asymptotic values. In these examples, $\rho = 1$ gives the right value.

In the next two experiments, we examine the behavior of the option price and free boundary for different values of Y and M. We conclude from figures 2 and 3 that the American option price is an increasing function of Y and a decreasing function of M. We mention that the results shown in Figure 2 are in accordance with the numerical tests in [17], Figure 6.

The last test is designed to verify the smooth-fit principle. According to [1], the smooth-fit principle holds for perpetual American put options in the bounded variation case considered here if and only if the drift $r - q + \omega$ is negative, or an additional condition on the jump measure is satisfied for zero drift. In Figure 4, left, we show the numerical derivative v_s at time $T = 1$, for a set of parameters giving negative drift. In this case we have smooth-fit. For a second set of parameters chosen such that the drift is positive, we see a discontinuous derivative in Figure 4, right, so there is no smooth-fit.

A Analytic formula for European option prices

We include here the analytic expression given in [16] for European options, adapted to the case of a CGMY process:

$$u(t,x) = \frac{e^{-rt}}{2\pi} \int_{i\alpha-\infty}^{i\alpha+\infty} \exp\left[-izx + t\phi_0(-z)\right] \hat{\psi}(z) dz, \tag{45}$$

where $\hat{\psi}(z)$ is the generalized Fourier transform of the payoff ψ, which for a put option is given by

$$\hat{\psi}(z) = -\frac{K^{iz+1}}{z^2 - iz}, \tag{46}$$

and the risk-neutral characteristic function ϕ_0 to be used is obtained by substituting μ by ω from (6) in expression (4), i.e.,

$$\begin{aligned} \phi_0(z) = (r - q + \omega)iz - \frac{\sigma^2}{2}z^2 \\ + C\Gamma(-Y)\left\{(M - iz)^Y - M^Y + (G + iz)^Y - G^Y\right\}. \end{aligned} \tag{47}$$

The constant α in (45) is determined by the region of validity of (46) together with the strip of regularity of (47). In this case we may pick $\alpha \in (-G, 0)$. A method using the FFT algorithm was proposed in [9] to evaluate an analogous version of (45).

References

[1] L. Alili and A. E. Kyprianou, *Some remarks on first passage of Lévy processes, the American put and pasting principles*. To appear in Annals of Applied Probability, 2004.

[2] A. Almendral and C. W. Oosterlee, *On American options under the Variance Gamma process*, tech. report, Delft University of Technology, 2004.

[3] A. Almendral and C. W. Oosterlee, *Numerical valuation of options with jumps in the underlying*, Appl. Numer. Math., 53 (2005), pp. 1–18.

[4] S. Asmussen and J. Rosiński, *Approximations of small jumps of Lévy processes with a view towards simulation*, J. Appl. Probab., 38 (2001), pp. 482–493.

[5] S. I. Boyarchenko and S. Z. Levendorskiĭ, *Non-Gaussian Merton-Black-Scholes theory*, vol. 9 of Advanced Series on Statistical Science & Applied Probability, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[6] M. J. Brennan and E. S. Schwartz, *The valuation of the American put option*, Journal of Finance, 32 (1977), pp. 449–462.

[7] P. Carr, H. Geman, D. B. Madan, and M. Yor, *Stochastic volatility for Lévy processes*, Math. Finance, 13 (2003), pp. 345–382.

[8] P. P. Carr, H. Geman, D. B. Madan, and M. Yor, *The fine structure of asset returns: An empirical investigation*, Journal of Business, 75 (2002), pp. 305–332.

[9] P. P. Carr and D. B. Madan, *Option valuation using the Fast Fourier Transform*, Journal of Computational Finance, 2 (1999), pp. 61–73.

[10] P. P. Carr, D. B. Madan, and E. C. Chang, *The Variance Gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

[11] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall, Boca Raton, FL, 2004.

[12] R. Cont and E. Voltchkova, *A finite difference scheme for option pricing in jump diffusion and exponential Lévy models*, Tech. Report 513, Centre de Mathématiques Appliquées, Palaiseau, France, 2003.

[13] A. Hirsa and D. B. Madan, *Pricing American Options Under Variance Gamma*, Journal of Computational Finance, 7 (2004).

[14] P. Jaillet, D. Lamberton, and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Applicandae Mathematicae, 21 (1990), pp. 263–289.

[15] S. G. Kou, *A jump diffusion model for option pricing*, Management Science, 48 (2002), pp. 1086–1101.

[16] A. L. Lewis, *A simple option formula for general jump-diffusion and other exponential Lévy processes.* Available at http://ssrn.com/abstract=282110, September 2001.

[17] A. M. Matache, P. A. Nitsche, and C. Schwab, *Wavelet Galerkin pricing of American options on Lévy driven assets*, working paper, ETH, Zürich, 2003.

[18] S. Raible, *Lévy Processes in Finance: Theory, Numerics, and Empirical Facts*, Ph.D. Thesis, Inst. für Mathematische Stochastik, Albert-Ludwigs-Universität Freiburg, Germany, 2000.

[19] K.-i. Sato, *Basic results on Lévy processes*, in Lévy processes, O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick, eds., Birkhäuser Boston, Boston, MA, 2001, pp. 3–37.

[20] W. Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley & Sons Ltd., 2003.

[21] C. Van Loan, *Computational frameworks for the fast Fourier transform*, vol. 10 of Frontiers in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

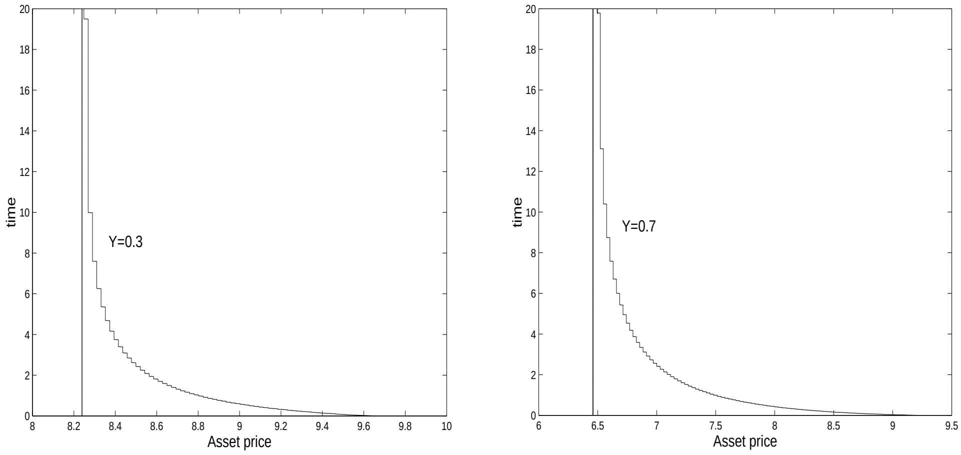

Figure 1: Exercise boundary and perpetual boundary for two different values of Y; $\sigma = 0$, $r = 0.1$, $q = 0$, $K = 10$, $T = 20$, $C = 1$, $G = 7$, $M = 9$.

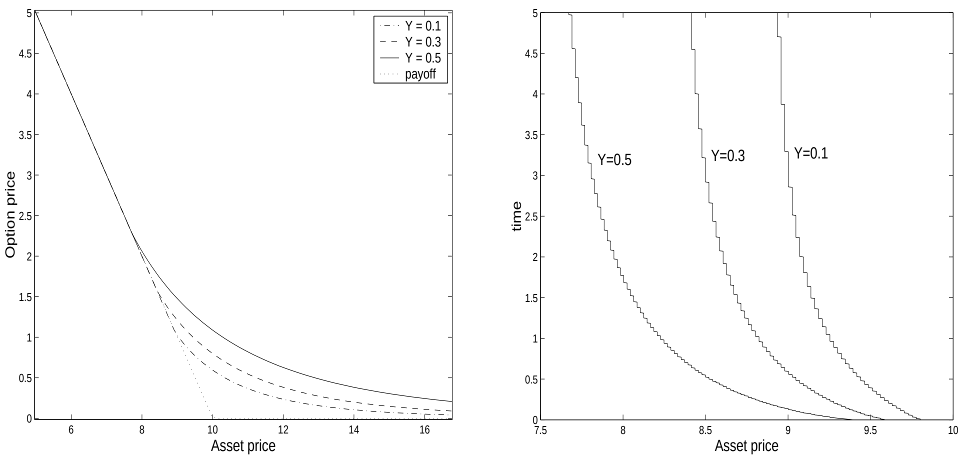

Figure 2: Left picture: Option prices for different values of parameter Y; $\sigma = 0$, $r = 0.1$, $q = 0$, $K = 10$, $T = 5$, $C = 1$, $G = 7.8$, $M = 8.2$. Right picture: Corresponding exercise boundaries.

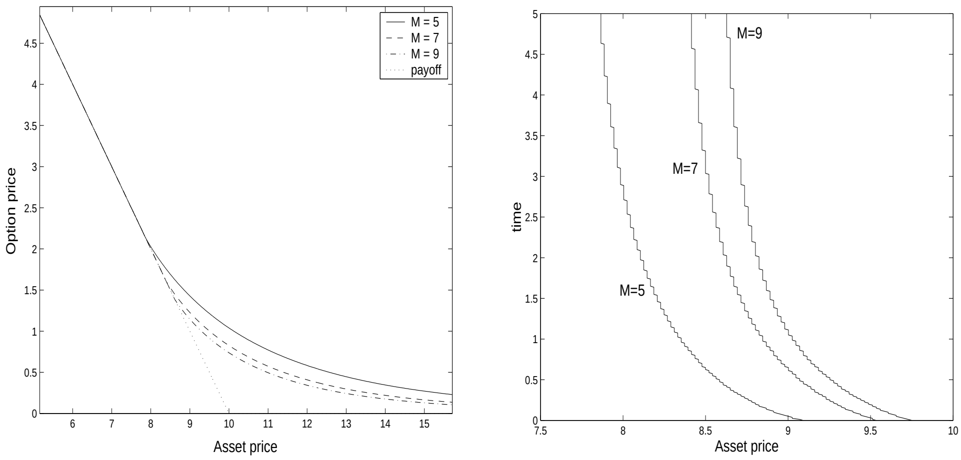

Figure 3: Left picture: Option prices for different values of parameter M; $\sigma = 0$, $r = 0.1$, $q = 0$, $K = 10$, $T = 5$, $C = 1$, $G = 7$, $Y = 0.2$. Right picture: Corresponding exercise boundaries.

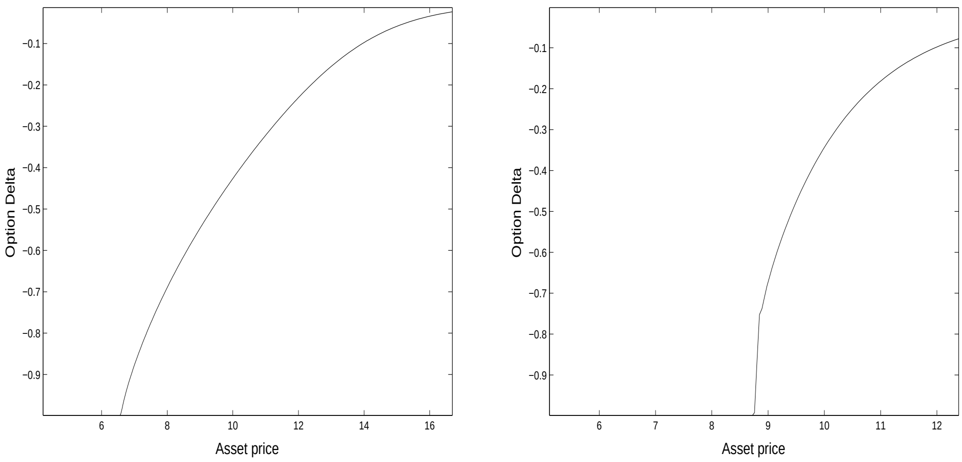

Figure 4: Left picture: Continuous option Delta for $G = 10$ and $M = 3$. Right picture: Discontinuous option Delta for $G = 7$ and $M = 9$; $\sigma = 0$, $r = 0.1$, $q = 0$, $K = 10$, $T = 1$ and $C = 1$.