

# Net Present Value with Uncertainty



**Note no**  
**Authors**

**SAMBA/05/06**  
**Xeni K. Dimakos**  
**Linda Reiersølmoen Neef**  
**Kjersti Aas**

**Date**

**1st February 2006**

## Norwegian Computing Center

Norsk Regnesentral (Norwegian Computing Center, NR) is a private, independent, non-profit foundation established in 1952. NR carries out contract research and development projects in the areas of information and communication technology and applied statistical modeling. The clients are a broad range of industrial, commercial and public service organizations in the national as well as the international market. Our scientific and technical capabilities are further developed in co-operation with The Research Council of Norway and key customers. The results of our projects may take the form of reports, software, prototypes, and short courses. A proof of the confidence and appreciation our clients have for us is given by the fact that most of our new contracts are signed with previous customers.

**Title** **Net Present Value with Uncertainty**

**Authors** **Xeni K. Dimakos** <Xeni.Dimakos@nr.no>  
**Linda Reiersølmoen Neef** <Linda.Neef@nr.no>  
**Kjersti Aas** <Kjersti.Aas@nr.no>

**Date** 1st February 2006

**Publication number** SAMBA/05/06

### **Abstract**

Projects may be evaluated and compared according to the Net Present Value (NPV) of their cash flow. The NPV is the discounted expected revenues minus the costs over the lifetime of the project. Traditional NPV calculations do not take into account the uncertainty in the variables that influence the revenues and costs.

We present a simulation approach to incorporate uncertainty in the NPV calculations. We build a joint stochastic model for the most influential input variables of the NPV calculation. From this model we simulate correlated scenarios of these variables and calculate the NPV. This way, the uncertainty in the input variables are transferred to the NPV calculations. The resulting probability distribution of the NPV takes into account worst-case scenarios that arise when several factors evolve in undesired directions at the same time. This allows us to measure the uncertainty associated with the NPV using confidence intervals or the Value-at-Risk (VaR).

**Keywords** Net present value, uncertainty, correlation, simulation, autoregressive models

**Target group** 1st ICEC & IPMA Global congress on project management

**Availability** Open

**Project** SIP-StAR

**Project number** 830105

**Research field** Technology, industry and administration

**Number of pages** 13

**© Copyright** Norwegian Computing Center

# 1 Introduction

Prior to initialisation, a project is often evaluated by calculating its Net Present Value (NPV). The NPV is defined as the discounted difference between the expected value of project revenues and costs over the lifetime of the project. NPV is the preferred criterion of profitability since it reflects the net contribution to the owner's equity considering his cost of capital. NPV calculations are typically used by the management to select between competing projects.

Traditional NPV calculations do not take into account the uncertainty in the risk factors that influence the NPV, for instance the selling price of the produced goods, exchange rates, salaries or maintenance costs. In order to study the uncertainty of the NPV, management have traditionally used approaches such as increasing the discount rate, comparing pessimistic and optimistic cash flows, or performing sensitivity analyses modifying one or several input quantities separately. Attempts to evaluate the uncertainty of the NPV are typically based on marginal considerations (e.g. Dailami et al. (1999)) or a categorisation of the association between risk factors (Warszawski and Sacks, 2004).

We propose a simulation approach to incorporate uncertainty in the NPV calculations. We construct a stochastic model that describes the simultaneous behaviour of the risk factors that have the largest influence on the uncertainty of the NPV of the project. The marginal models and relationship between risk factors are estimated based on historical time series data. By simulating from this model, we obtain the probability distribution of these risk factors over the time horizon of the project. When these distributions are incorporated in the NPV calculations, we obtain the probability distribution of the NPV. The results show the expected profitability and the risk that the profitability deviates from the expected value. As the full distribution of the NPV is available, we may select measures of uncertainty such as a confidence interval or Value-at-Risk (VaR), see Jorion (1997).

An essential feature of our approach is that we may choose to specify the level around which the simulations fluctuate. Historical data is used to estimate the uncertainty of the risk factors. The mean level may also be obtained from the historical data, or it can be consistent with a certain prespecified scenario level. For many risk factors this is an advantage as the historical data will not be representative of future development. If the risk factor is traded in a market, there might exist market forward curves that are far more relevant than historical levels.

The proposed methodology has been developed in co-operation with Boliden Odda AS and has been applied to evaluate their P2007 Expansion project. In this paper, this project is used to illustrate some of the model aspects. However, due to confidentiality requirements, the main results for the NPV are not included.

The main product of Boliden Odda AS is zinc. In their case the stochastic model included seven risk factors on which Boliden Odda has no influence since they are given by the market. The risk factors that were identified as crucial for the uncertainty of the NPV included commodity prices (the LME zinc price and the sulphuric acid price), a power spot price, two different exchange rates, as well as two other prices related to the processing of zinc.

The paper is organised as follows. Section 2 presents the framework and system requirements of our approach. In Section 3 we present the joint stochastic model for the selected risk factors. In Section 4 we explain how the model is fitted to historical data, while Section 5 shows how we simulate from the model. Section 6 demonstrates the quantities and results that can be obtained by applying the suggested approach. Finally, in Section 7 we summarise and discuss improvements of our approach.

## 2 Setting

### 2.1 Net present value

Our approach is motivated by the P2007 expansion project of Boliden Odda AS. Boliden Odda AS had developed a system based on several Microsoft Excel worksheets in which the calculation of the NPV of the project was one functionality. In the following we will assume that a similar system for the calculation of the NPV of the project of interest is available. This system calculates the value of the project  $V_t$  for each year  $t$  in the project period  $t = 1, \dots, T$ . Typically, the value of the project is a complex function of a large number of risk factors that influence the revenues  $R_t$  and costs  $C_t$  each year. Each risk factor is assessed by a yearly value that represents the level of the risk factor in the particular year. For simplicity, we assume that the risk factors that influence the revenues and costs are the same for each year in the project period. The value of the project in year  $t$  is then given by

$$\begin{aligned} V_t &= V_t(X_{1t}, \dots, X_{Nt}) \\ &= R_t(X_{1t}, \dots, X_{Nt}) - C_t(X_{1t}, \dots, X_{Nt}), \end{aligned}$$

where  $X_{1t}, \dots, X_{Nt}$  denote the risk factors that influence the revenues and costs. Depending of the functional form of  $R_t$  and  $C_t$  some risk factors might influence the revenues or the costs only. Applying yearly discount rates  $r_1, \dots, r_T$  the NPV

is given by

$$NPV = NPV(V_1, \dots, V_T) \\ = \sum_{t=1}^T \frac{V_t}{\prod_{j=1}^t (1 + r_j)}$$

Often a constant discount rate is applied, in which case  $r_t$  is replaced by  $r$  in the formulae above. Investment costs (CAPEX) are included by subtracting a constant term from the NPV.

## 2.2 System

In the following we assume that risk factors  $S_{1t}, \dots, S_{Mt}$  that influence either the revenues or the costs, or both, has been identified as crucial for the NPV calculation of the project of interest. Typically, these risk factors are a subset of all the risk factors included in the NPV calculation. To simplify the notation, we assume that the identified factors are the first  $M$  risk factors. Hence,  $X_{1t}, \dots, X_{Nt} = S_{1t}, \dots, S_{Mt}, X_{(M+1)t}, \dots, X_{Nt}$ .

Our model produces simulations  $S_{1t}^{*b}, \dots, S_{Mt}^{*b}, b = 1, \dots, B; t = 1, \dots, T$ , where  $B$  denotes the number of simulations. These simulations represent the probability distribution of the selected risk factors and hence all the information about the uncertainty. Simulated values of the NPV are obtained by performing the NPV calculation using the simulated values, so that

$$NPV^{*b} = NPV(V_1^{*b}, \dots, V_T^{*b}), b = 1, \dots, B,$$

where

$$V_t^{*b} = R_t(S_{1t}^{*b}, \dots, S_{Mt}^{*b}, X_{(M+1)t}, \dots, X_{Nt}) - C_t(S_{1t}^{*b}, \dots, S_{Mt}^{*b}, X_{(M+1)t}, \dots, X_{Nt}).$$

In order to integrate the NPV calculation and the model of Section 3, it is required that the selected risk factors are changeable variables in the system. Typically, the simulated risk factors are written to file, and then successively feed to the system calculating the NPV. For each simulation of the risk factors, the corresponding NPV value is recorded.

In the case of Boliden Odda AS, the values  $S_{1t}, \dots, S_{Mt}, t = 1, \dots, T$ , were specified as a matrix in one of the sheets in the Excel workbook. The Excel workbook was extended with a macro written in Microsoft Visual Basic 6.3. The purpose of the macro was to read the simulated values from the files and for each simulation activate Boliden's NPV worksheet, and calculate the NPV value. The macro runs through all the simulations and writes the resulting NPVs to a new Excel worksheet.

### 3 Model

In our model, all the risk factors are treated on a logarithmic scale. To simplify the notation, we let  $S_t$  denote the value at time  $t$  of one of the selected risk factors. The log-values are modelled as a sum of a deterministic trend and a stochastic residual process,

$$s_t = \log(S_t) = \mu_t + r_t.$$

Here  $\mu_t$  describes the trend of the time series. We will assume that the trend is a linear function of time, that is,

$$\mu_t = a + bt.$$

The residual process  $r_t$  models deviations from the trend, that is the variability in the risk factors that may not be described by the trend. The model selected for  $r_t$  needs to reflect the characteristics of the variability of the risk factor. In the case of Boliden Odda AS, an autoregressive assumption was reasonable for the selected risk factors. An autoregressive assumption implies that we expect the time series of the risk factors to be mean reverting, i.e. to have a tendency to return to some normal level. In statistical terms our model is stationary and there exists a mean-level around which the process fluctuates. The simplest possible stationary model is an autoregressive (AR) model of order one, which we denote by AR(1). We model the residual process as such an AR(1) process,

$$r_t = \alpha r_{t-1} + \varepsilon_t.$$

Here  $0 < \alpha < 1$  is the AR-parameter, and  $\varepsilon_t$  denotes the innovations. The AR-parameter determines the mean-reversion rate of the series. A small  $\alpha$  gives a strongly mean-reverting series, while an  $\alpha$  close to one gives a series that tends to return very slowly to the mean level.

If the selected risk factors were stocks, a random-walk or a GARCH-model (Bollerslev, 1986) are more realistic than the autoregressive model. A random-walk is obtained by simply letting  $\alpha = 1$ .

For all the risk factors in our model, we assume that the innovations follow a normal distribution with mean 0 and standard deviation  $\sigma$ . This assumption needs to be verified using the data at hand.

In order to build a simultaneous model for all the risk factors, we need to incorporate the correlations between them in the model. We use a multivariate normal distribution for the innovation processes of all the risk factors. This multivariate normal distribution has a zero mean vector and a correlation matrix  $\mathbf{R}$ .

To summarise, each risk factor of the model is defined by the parameter vectors  $a$ ,  $b$ ,  $\alpha$  and  $\sigma$ , and the relationship between the risk factors by the correlation

matrix  $\mathbf{R}$  of the multivariate normal distribution of the innovations. With  $M$  risk factors there are  $4M + M(M - 1)/2$  parameters in the model.

## 4 Estimation

### 4.1 Historical data

The model parameters  $a$ ,  $b$ ,  $\alpha$ ,  $\sigma$  and  $\mathbf{R}$  are estimated using historical time series data. For each risk factor historical values must be collected from a relevant time period. The source and availability of the data will depend on the risk factor, typically stocks and commodity exchanges are sources of information. The time horizon of NPV calculations is usually 10 to 20 years. It is recommendable to use time series that includes several time periods of this length, but in practice this is rarely possible. Also, as economic regimes changes through time, older data tends to be less representative of current and future regimes. Therefore, we collect the longest, obtainable and relevant time series.

In most NPV calculations, a yearly time resolution is applied. However, it is possible to use data on a finer resolution such as daily or monthly. In this case, the model of Section 3 also needs to have the same, finer, resolution. The daily or monthly simulations from the model are aggregated to yearly notations, for instance by averaging.

### 4.2 Procedure

The model from Section 3 is fitted to historical data as follows:

1. Compute the logarithm of historical data,  $\log(S_t)$ .
2. Estimate the coefficients  $a$  and  $b$  of the trend by regressing  $\log(S_t)$  on  $t$ . This yields an estimated trend  $\hat{\mu}_t = \hat{a} + \hat{b}t$ .
3. Find the estimated residual process  $\hat{r}_t = \log(S_t) - \hat{\mu}_t$ .
4. Estimate the autoregressive parameter  $\alpha$  of the residual process and the standard deviation  $\sigma$ . This is common functionality in most statistical software packages. We used the ar-function of S-Plus, Version 6.2.1.
5. Calculate the estimated innovations  $\hat{\varepsilon}_t = \hat{r}_t - \hat{\alpha}\hat{r}_{t-1}$ .
6. Estimate the correlation matrix of the innovations. If the time series have different lengths, correlations can be estimated pairwise, using the maximum available length for each pair. The resulting correlation matrix need not be positive definite, in which case it can be transformed to one that is by using the method of Rebonato and Jäckel (1999).

# 5 Simulation

## 5.1 The risk factors

The  $b$ th simulation from the fitted model is generated as follows:

1. Simulate the innovations from the multivariate normal distribution with correlation matrix  $\mathbf{R}$  for each of the  $T$  years in the NPV time horizon. For simplicity, we suppress the risk factor index in the notation, and let  $\varepsilon_1^{*b}, \dots, \varepsilon_T^{*b}$  denote these innovations for one of the risk factors.
2. Generate the autoregressive processes for each of the risk factors as  $r_t^{*b} = \hat{\alpha}r_{t-1}^{*b} + \varepsilon_t^{*b}$ ,  $t = 1, \dots, T$ .
3. Add the AR-processes to the trends, using the mean of the historical data as the future trend,  $s_t^{*b} = \hat{a} + \hat{b}t + r_t^{*b}$ ,  $t = 1, \dots, T$ .
4. Transform to the original scale,  $S_t^{*b} = \exp(s_t^{*b})$ ,  $t = 1, \dots, T$ .

The above procedure is repeated for a certain number of simulations  $b = 1, \dots, B$ . If the time resolution of the model and the NPV calculation differs, the simulations need to be aggregated to a yearly resolution.

By selecting the acceptable standard error of quantiles of the resulting NPV distribution, the number of simulations can be determined by applying an approximation (Jorion, 1997, p. 99).

Figure 1 shows an illustration from the case of Boliden Odda AS. The historical LME Zinc data has a monthly resolution and notations from January 1980 to December 2005. The monthly simulations cover the 18 year NPV time horizon, starting in January 2006. The figure shows three examples of simulated paths for this risk factor.

## 5.2 Adjusting the simulations

The simulations  $S_t^{*b}$ ,  $t = 1, \dots, T$ ,  $b = 1, \dots, B$  for one or several of the risk factors, may be adjusted so that their mean level is consistent with a certain prespecified scenario level, rather than the estimated historical level  $\hat{\mu}_t$ . Such a level can be based on the managements long term expectation or it can be based forward curves defined by a market. The simulations are adjusted by multiplying with the ratio between the required level and the average of the simulations over the full time horizon for each year separately.

### LME zinc price Historical and examples of simulated values

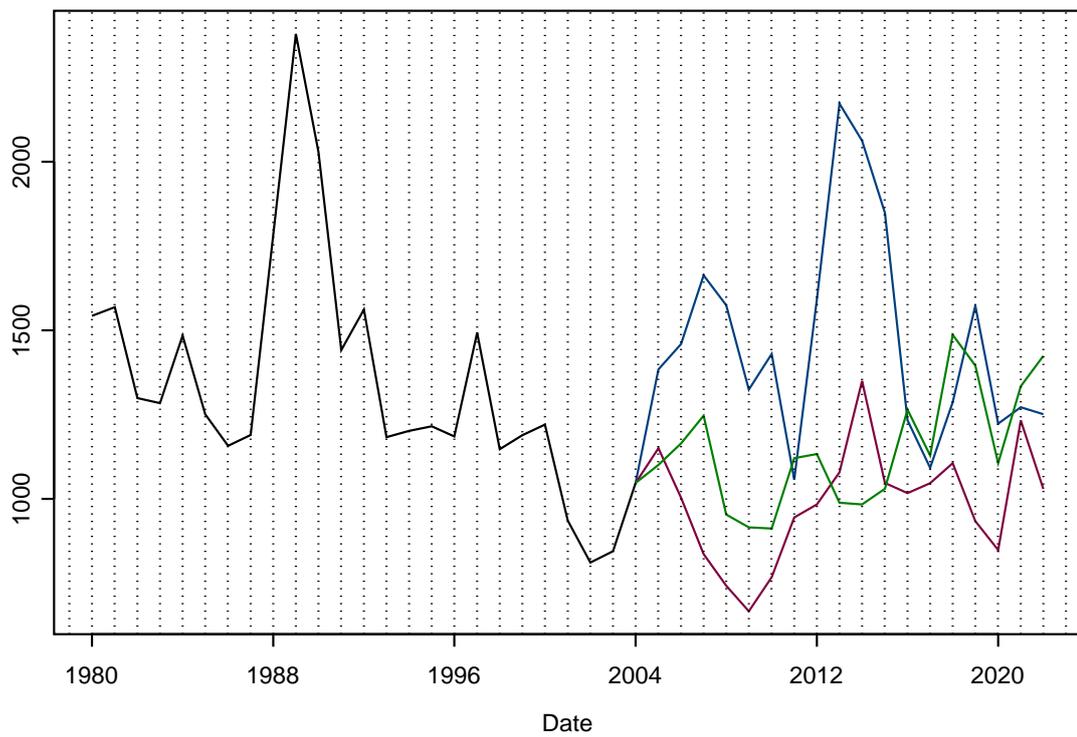


Figure 1. Illustration of the historical data for the LME Zinc price (black line) and three different simulated paths, on a monthly time resolution, for a time horizon of 18 years.

## 6 Results

Once the simulations  $NPV^{*b}$ ,  $b = 1, \dots, B$  have been obtained they can be used to estimate the distribution of the NPV. A histogram or density-estimate can be found by feeding the simulations to standard statistical software functions. Figure 2 is an illustration of the type of results we are able to obtain. The average value, standard deviation and empirical quantiles of the simulations approximate the expectation, theoretical standard deviation and the confidence interval, respectively. Any measure based on the true probability distribution may be estimated using the empirical counterpart.

In a comparison between projects, the information about the uncertainty which is obtained using the outlined approach, is valuable. Two competing projects may have quite similar NPVs, but their uncertainty may differ greatly. If risk minimisation is an aim, management could prefer the project with lower expected NPV, but reduced uncertainty. Worst-case scenarios are another aspect of risk management, for which the lower quantiles of the estimated NPV distribution represent important information.

## 7 Conclusions and further work

In this paper we have presented an approach to incorporate uncertainty in a NPV calculation. Stochastic simulations of risk factors known to influence the NPV are generated from a joint stochastic model. A correlation matrix represents the relationship between the risk factors. The presence of positive correlations between risk factors that influence the NPV in the same direction is captured in the NPV calculations. This implies that we neither underestimate the up-or downside as done by assuming independence, nor are we strictly conservative as done when assuming linear relationships.

Our approach does not account for all the uncertainty associated with the NPV. Only a subset of all the risk factors is included, and the selection of risk factors is based on subjective judgement. By increasing the number of risk factors in the model, we account for more of the uncertainty, but increase the complexity of the model. Also, we do not account for the uncertainty associated with the choice of model. Sensitivity towards the model choice, may be evaluated by fitting several different models.

The methodological challenges of our approach are associated with designing the marginal and joint stochastic model of the risk factors of interest. The selected risk factors may be very different in nature, in which case the joint model can not be a standard multivariate model. Such marginal distributions may be joined in a

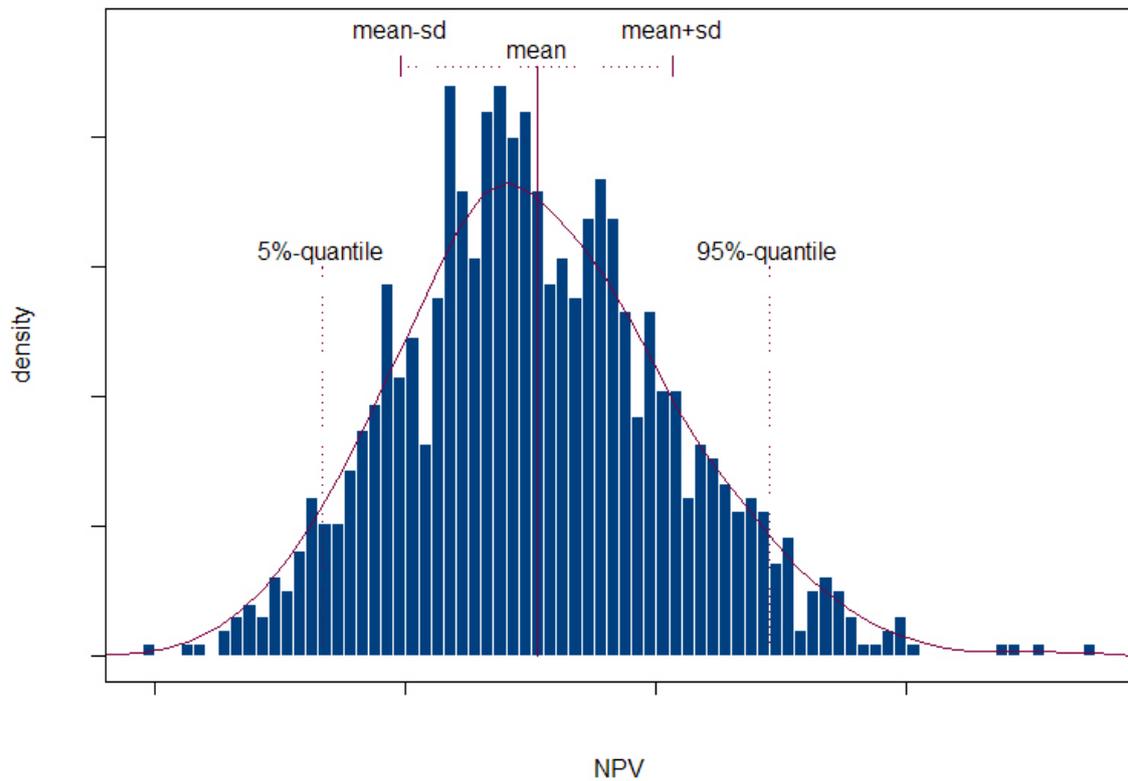


Figure 2. Illustration of a simulated probability distribution of the NPV, using 1000 simulations. The distribution may be used to find a 90% confidence interval, the mean or the standard deviation, as indicated in the figure.

multivariate model by using copulas Embrechts et al. (2001). Also, we suggest a simple AR(1) model. An alternative is the class of Vector Autoregressive Models (VAR), see Sims (1980), that allows for more flexible dependency structures.

The time horizon of the NPV calculations presents another challenge. It is often difficult to find relevant historical time series data for the risk factors of interest of a similar length. When the available time series are too short we stand the risk of underestimating the extremes of the risk factors. The time aspect makes it important to perform a subjective assessment of the shortcoming of the data at hand, and if possible, correct for this.

## References

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31:307–327.

Dailami, M., Lipkovich, I., and Dyck, J. V. (1999). INFRISK: A computer simulation approach to risk management in infrastructure project finance transactions.

Embrechts, P., McNeil, A. J., and Straumann, D. (2001). Correlation and dependency in risk management: Properties and pitfalls. In *Value at Risk and Beyond*. Cambridge University Press.

Jorion, P. (1997). *Value at Risk*. McGraw-Hill.

Rebonato, R. and Jäckel, P. (1999). The most general methodology to create a valid correlation matrix for risk management and option pricing purposes. Technical report, Quantitative Research Centre of the NatWest Group.

Sims, C. A. (1980). Macroeconomics and reality. *Econometrica*, 48:1–46.

Warszawski, A. and Sacks, R. (2004). Practical multifactor approach to evaluating risk of investments in engineering projects. *Journal of Construction Engineering and Management*, 130(3):357–367.