Penalty methods for the numerical solution of American multi-asset option problems.

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Abstract

We derive and analyse a penalty method for solving American multi-asset option problems. A small, non-linear penalty term is added to the Black-Scholes equation. This approach gives a fixed solution domain, removing the free and moving boundary imposed by the early exercise feature of the contract. Explicit, implicit and semi-implicit finite difference schemes are derived, and in the case of independent assets, we prove that the approximate option prices satisfy some basic properties of the American option problem. Several numerical experiments are carried out in order to investigate the performance of the schemes. We give examples indicating that our results are sharp. Finally, experiments indicate that in the case of correlated underlying assets, the same properties are valid as in the independent case.

1 Introduction

American derivatives are popular trading instruments in todays financial markets. We consider American put options where the payoff depends on more than one underlying. Such option prices can be modelled by higher dimensional generalisations of the original Black-Scholes equation [2]. The purpose of this paper is to extend the penalty method discussed in [11] to multi-asset American put option problems.

Various numerical techniques can be applied to price multi-variate derivatives. Higher dimensional generalisations of lattice binomial methods can be used, c.f. [3], where European options based on three underlying options

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are solved numerically. Another way of pricing multi-asset derivatives is by Monte-Carlo simulation techniques, c.f. [1]. In a wide range of scientific fields, finite element and finite volume methods (FEM and FDM) are popular. For studies of FEM and FDM for numerical valuation of financial derivatives, c.f. [17, 8, 5]. Finite difference methods are also commonly used for solving the Black-Scholes equation in higher dimensions, c.f. [15] for a study of the singularity-separating method for two factor models, utilising a finite difference approach.

The idea behind the penalty method for multi-asset option models is similar to the method described in [11]. American put options can be exercised at any time before expiry. This introduces a free and moving boundary problem. By adding a certain penalty term to the Black-Scholes equation, we extend the solution to a fixed domain. As the solution approaches the payoff function at expiry, the penalty term forces the solution to stay above it. When the solution is far from the barrier, the term is small and thus the Black-Scholes equation is approximatively satisfied in this region.

A similar approach was introduced by Forsyth and Vetzal in [16] for American options with stochastic volatility. In their work they add a source term to the discrete equations. Our method represents a refinement of their work in the sense that the penalty term is added to the continuous equation. For independent underlying assets, this leads to restrictions regarding the magnitude of the penalty term as well as conditions for the discretization parameters. Also, by choosing a semi-implicit finite difference discretization, we avoid solving nonlinear algebraic equations and thereby enhance the overall computational efficiency.

We present numerical experiments illustrating the properties of the schemes. In the case of correlated underlying assets, we have been unable to derive proper bounds on the numerical solutions. However, numerical experiments indicate that similar properties are present in such cases.

This paper is organised as follows: In Section 2 we describe the multi-asset Black-Scholes equation, together with the penalty formulation of the problem. The boundary conditions corresponding to zero values of the underlying assets are obtained by solving lower dimensional Black-Scholes equations. In Section 4, numerical schemes for the two-factor model problem are derived, starting by specifying the two-factor model problem. First, an explicit scheme is presented, and then both a semi-implicit and a fully implicit scheme are defined. Analysis of these schemes are carried out in Section 4, under the assumption that the underlying assets are independent. Restrictions regarding the time step size and the penalty term are then provided for all three schemes. In the last section of this paper, we present a series of numerical experiments, starting by comparing the fully implicit and the semi-implicit scheme with respect to computational efficiency. In Section 5, we show that numerical experiments indicate that for our model data, the restrictions derived in Section 4 for independent assets are valid also when the underlying assets are correlated. Finally, we make some conclusive remarks in Section 6.

2 American multi-asset option problems

The multi-dimensional version of the Black-Scholes equation takes the form

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial P}{\partial S_i} - rP = 0,$$

see e.g. [7], [10] or [13]. Here, P is the value of the contract, S_i is the value of the ith underlying asset, n is the number of underlying assets, $\rho_{i,j}$ is the correlation between asset i and asset j, r is the risk free interest rate and D_i is the dividend yield paid by the ith asset.

For a majority of multi-asset option models the payoff function at expiry can be written on the form

$$\phi(S_1, \dots, S_n) = \max\left(E - \sum_{i=1}^{n} \alpha_i S_i, 0\right), \tag{1}$$

where E and $\alpha_1, \dots, \alpha_n$ are given constants, see [10]. We will in this paper consider put options, i.e.

$$E, \alpha_1, \dots, \alpha_n \geq 0.$$

Notice that the American early exercise feature of the contract imposes the constraint

$$P(S_1, \dots, S_n, t) \geq \phi(S_1, \dots, S_n)$$

on the solution for all admissible values of $S_1, \dots, S_n$ and t.

In the case of American options the solution domain can be divided into two parts. In one region the price of the option satisfies the Black-Scholes equation and in the second subdomain it equals the payoff function ϕ. This leads to the linear complementarity form of the problem. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial}{\partial S_i} - r,$$

and

$$\begin{aligned} \Omega &= \{(S_1, \dots, S_n);\ S_j > 0 \text{ for } j = 1, \dots, n\} = \mathbb{R}_+^n, \\ \Omega_i &= \{(S_1, \dots, S_{i-1}, 0, S_{i+1}, \dots, S_n);\ S_j \geq 0 \text{ for } j \neq i\}, \\ \mathbf{S} &= (S_1, \dots, S_n). \end{aligned}$$

If T represents the time of expiration of the contract, then the American put problem can be written on the form

$$(P-\phi)\,\mathcal{L}P = 0 \quad \text{in } \Omega\times[0,T], \tag{2}$$
$$\mathcal{L}P \le 0 \quad \text{in } \Omega\times[0,T], \tag{3}$$
$$P(\mathbf{S},t) \ge \phi(\mathbf{S}) \quad \text{in } \Omega\times[0,T], \tag{4}$$
$$P(\mathbf{S},T) = \phi(\mathbf{S}) \quad \text{for all } \mathbf{S}\in\Omega, \tag{5}$$
$$P(\mathbf{S},t) = g_i(\mathbf{S},t) \quad \text{for all } \mathbf{S}\in\Omega_i\times[0,T] \text{ and } i=1,\dots,n, \tag{6}$$
$$\lim_{S_i\to\infty} P(\mathbf{S},t) = G_i(S_1,\dots,S_{i-1},S_{i+1},\dots,S_n,t) \tag{7}$$
$$\text{for all } \mathbf{S}\in\Omega\times[0,T] \text{ and } i=1,\dots,n,$$

and both P and its first derivatives must be continuous. Here $g_i(\cdot,\cdot)$ and $G_i(\cdot,\cdot)$ are given functions providing suitable boundary conditions. Typically, $g_i(\cdot,\cdot)$ is determined by solving the associated $n-1$ dimensional American put problem and $G_i(\cdot,\cdot)$ is identical to zero. Further details can be found in Section 5. Until then, we will assume that the boundary conditions are consistent with the constraint imposed by the early exercise feature of the option, i.e. that $g_i(\cdot,\cdot)$ and $G_i(\cdot,\cdot)$ are consistent with the constraint (4).

2.1 A penalty method

Define the barrier function,

$$q(S_1,\dots,S_n) = E - \sum_{i=1}^{n} \alpha_i S_i.$$

As for American single-asset option problems, cf. [11], a penalty method for solving (2)-(7) can defined as follows

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-D_i)S_i\frac{\partial P}{\partial S_i} - rP \tag{8}$$
$$+\frac{\epsilon C}{P+\epsilon-q} = 0, \quad \mathbf{S}\in\Omega,\ t\in[0,T],$$
$$P(\mathbf{S},T) = \phi(\mathbf{S}) \quad \text{for all } \mathbf{S}\in\Omega, \tag{9}$$
$$P(\mathbf{S},t) = g_i(\mathbf{S},t) \quad \text{for all } \mathbf{S}\in\Omega_i\times[0,T] \text{ and } i=1,\dots,n, \tag{10}$$
$$\lim_{S_i\to\infty} P(\mathbf{S},t) = G_i(S_1,\dots,S_{i-1},S_{i+1},\dots,S_n,t) \tag{11}$$
$$\text{for all } \mathbf{S}\in\Omega\times[0,T] \text{ and } i=1,\dots,n,$$

where $0<\epsilon\ll 1$ is a small parameter and C is a positive constant. Note that the penalty term

$$\frac{\epsilon C}{P+\epsilon-q}$$

is of order ϵ in regions where $P(\mathbf{S}, t) \gg q(\mathbf{S})$, and hence the Black-Scholes equation is approximately satisfied. On the other hand, as P approaches q this term is approximately equal to C assuring that the early exercise constraint (4) is not violated. In Section 4 we will prove that for a two-factor problem, with independent assets, a discrete analogue to (4) holds provided that $C \geq rE$.

3 Discretization

For the sake of simplicity we will define our numerical schemes for a two-factor model problem and use x and y, instead of the more conventional notation S_1 and S_2, to represent the asset prices. The numerical methods and analysis presented in this paper can easily be extended to general n-dimensional American option problems, provided that the payoff function at expiry is on the form (1).

3.1 A two-factor model problem

We will consider the following penalty formulation of an American put problem with two underlying assets, i.e. $n = 2$,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 P}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 P}{\partial x \partial y} \tag{12}$$

$$+(r - D_1)x\frac{\partial P}{\partial x} + (r - D_2)y\frac{\partial P}{\partial y} - rP + \frac{\epsilon C}{P + \epsilon - q} = 0, \; x, y > 0, \; t \in [0, T),$$

$$P(x, y, T) = \phi(x, y), \quad x, y \geq 0, \tag{13}$$

$$P(x, 0, t) = g_1(x, t), \quad x \geq 0, \; t \in [0, T], \tag{14}$$

$$P(0, y, t) = g_2(y, t), \quad y \geq 0, \; t \in [0, T], \tag{15}$$

$$\lim_{x \to \infty} P(x, y, t) = G_1(y, t), \quad y \geq 0, \; t \in [0, T], \tag{16}$$

$$\lim_{y \to \infty} P(x, y, t) = G_2(x, t), \quad x \geq 0, \; t \in [0, T], \tag{17}$$

where

$$q(x, y) = E - (\alpha_1 x + \alpha_2 y), \quad \phi(x, y) = \max\left(q(x, y), 0\right). \tag{18}$$

Let, for given positive integers I, J and N,

$$\Delta x = \frac{x_\infty}{I + 1}, \quad \Delta y = \frac{y_\infty}{J + 1}, \quad \Delta t = \frac{T}{N + 1}, \tag{19}$$

$$x_i = i\Delta x, \quad i = 0, \ldots, I + 1, \tag{20}$$

$$y_j = j\Delta y, \quad j = 0, \ldots, J + 1, \tag{21}$$

$$t_n = n\Delta t, \quad n = 0, \ldots, N + 1, \tag{22}$$

$$q_{i,j} = q(x_i, y_j), \quad i = 0, \ldots, I + 1 \text{ and } j = 0, \ldots, J + 1, \tag{23}$$

$$P_{i,j}^n \approx P(x_i, y_j, t_n). \tag{24}$$

Here x_∞ and y_∞ are the upper boundaries of the truncated solution domain. Throughout this paper we will assume that $\Delta x = \Delta y = h$.

The discrete final condition and boundary conditions are defined in a straight forward manner

$$
\begin{aligned}
P_{i,j}^{N+1} &= \max(q_{i,j}, 0), \quad i = 0, \dots, I+1 \text{ and } j = 0, \dots, J+1, && (25)\\
P_{i,0}^{n} &= (g_1)_i^n, \quad i = 0, \dots, I+1 \text{ and } n = 0, \dots, N+1, && (26)\\
P_{0,j}^{n} &= (g_2)_j^n, \quad j = 0, \dots, J+1 \text{ and } n = 0, \dots, N+1, && (27)\\
P_{i,J+1}^{n} &= (G_1)_i^n, \quad i = 0, \dots, I+1 \text{ and } n = 0, \dots, N+1, && (28)\\
P_{I+1,j}^{n} &= (G_2)_j^n, \quad j = 0, \dots, J+1 \text{ and } n = 0, \dots, N+1. && (29)
\end{aligned}
$$

Here $(g_1)_i^n$, $(g_2)_j^n$, $(G_1)_i^n$, $(G_2)_j^n$ are discrete approximations of $g_1(x_i, t_n)$, $g_2(y_j, t_n)$, $G_1(x_i, t_n)$, $G_2(y_j, t_n)$, respectively. We let $(G_1)_i^n = (G_2)_j^n = 0$, whereas $(g_1)_i^n$ and $(g_2)_j^n$ are obtained by solving the corresponding one-dimensional Black-Scholes equations.

In order to simplify the notation needed in this paper we introduce the finite difference operators

$$
\begin{aligned}
\mathcal{D}_{xx} Q_{i,j}^n &= \frac{Q_{i+1,j}^n - 2Q_{i,j}^n + Q_{i-1,j}^n}{h^2}, \quad \mathcal{D}_{yy} Q_{i,j}^n = \frac{Q_{i,j+1}^n - 2Q_{i,j}^n + Q_{i,j-1}^n}{h^2}, && (30)\\
\mathcal{D}_{xy} Q_{i,j}^n &= \frac{Q_{i+1,j+1}^n - Q_{i,j+1}^n - Q_{i+1,j}^n + 2Q_{i,j}^n - Q_{i-1,j}^n - Q_{i,j-1}^n + Q_{i-1,j-1}^n}{2h^2}, && (31)\\
\mathcal{D}_{x} Q_{i,j}^n &= \frac{Q_{i+1,j}^n - Q_{i,j}^n}{h}, \quad \mathcal{D}_{y} Q_{i,j}^n = \frac{Q_{i,j+1}^n - Q_{i,j}^n}{h}, && (32)\\
\mathcal{D}_{t} Q_{i,j}^n &= \frac{Q_{i,j}^n - Q_{i,j}^{n-1}}{\Delta t}, && (33)
\end{aligned}
$$

where $\{Q_{i,j}^n\}_{i,j=0}^{I+1,J+1}$, for $n = 0, \dots, N+1$, is a discrete function defined on the mesh defined in equations (19)-(22). Since we use upwind differences in (32), and a first order approximation of the time derivative in (33), the truncation error of the resulting scheme is $\mathcal{O}(h, \Delta t)$. Throughout this paper we will assume[1] that

$$
r \geq D_1,\, D_2,
$$

and, hence we use an upwind differencing to discretize the transport terms in (12), cf. (32).

3.2 An explicit scheme

Assume we know the solution at time step n, and that we wish to to compute P^{n-1}. Applying the space and time finite difference operators at time step

[1] If $r \leq D_1$, or $r \leq D_2$ we preserve upwind differencing by replacing (32) with the proper finite difference operator.

n, the explicit scheme reads

$$\mathcal{D}_t P_{i,j}^n + \frac{1}{2}\sigma_1^2 x_i^2 \ \mathcal{D}_{xx} P_{i,j}^n + \frac{1}{2}\sigma_2^2 y_j^2 \ \mathcal{D}_{yy} P_{i,j}^n + \rho\sigma_1\sigma_2 x_i y_j \ \mathcal{D}_{xy} P_{i,j}^n$$
$$+(r - D_1)x_i \ \mathcal{D}_x P_{i,j}^n + (r - D_2)y_j \ \mathcal{D}_y P_{i,j}^n - rP_{i,j}^n$$
$$+\frac{\epsilon C}{P_{i,j}^n + \epsilon - q_{i,j}} = 0,$$

for $i = 1, \dots, I$, $j = 1, \dots, J$ and $n = N + 1, N, \dots, 1$. The final condition and boundary conditions are defined in (25)-(29).

Defining

$$\begin{aligned} F(V_1, V_2, V_3, V_4, V_5, V_6, V_7, q, x, y) = \\ e(x,y)\, V_1 + [b(y) - e(x,y)]\, V_2 + [a(x) - e(x,y)]\, V_3 \\ +[1 - 2a(x) - 2b(y) + 2e(x,y) - c(x) - d(y) - r\Delta t]\, V_4 \\ +[a(x) - e(x,y) + c(x)]\, V_5 + [b(y) - e(x,y) + d(y)]\, V_6 + e(x,y)\, V_7 \\ +\frac{\epsilon C \Delta t}{V_4 + \epsilon - q}, \end{aligned} \tag{34}$$

where

$$\begin{aligned} a(x) = \frac{1}{2}\frac{\Delta t}{h^2}\sigma_1^2 x^2, \quad b(y) = \frac{1}{2}\frac{\Delta t}{h^2}\sigma_2^2 y^2, \quad c(x) = (r - D_1)\frac{\Delta t}{h} x, \\ d(y) = (r - D_2)\frac{\Delta t}{h} y, \quad e(x,y) = \frac{1}{2}\frac{\Delta t}{h^2}\rho\sigma_1\sigma_2 xy, \end{aligned} \tag{35}$$

this scheme can be written on the form

$$P_{i,j}^{n-1} = F(P_{i-1,j-1}^n, P_{i,j-1}^n, P_{i-1,j}^n, P_{i,j}^n, P_{i+1,j}^n, P_{i,j+1}^n, P_{i+1,j+1}^n, q_{i,j}, x_i, y_j). \tag{36}$$

3.3 Semi-implicit and fully implicit schemes

The implicit and semi-implicit method are obtained by applying the spatial finite difference operators at time step n and the time difference at time step $n + 1$,

$$\begin{aligned} \mathcal{D}_t P_{i,j}^{n+1} + \frac{1}{2}\sigma_1^2 x_i^2 \ \mathcal{D}_{xx} P_{i,j}^n + \frac{1}{2}\sigma_2^2 y_j^2 \ \mathcal{D}_{yy} P_{i,j}^n + \rho\sigma_1\sigma_2 x_i y_j \ \mathcal{D}_{xy} P_{i,j}^n \\ +(r - D_1)x_i \ \mathcal{D}_x P_{i,j}^n + (r - D_2)y_j \ \mathcal{D}_y P_{i,j}^n - rP_{i,j}^n \\ +\frac{\epsilon C}{P_{i,j}^{n+1/2} + \epsilon - q_{i,j}} = 0, \end{aligned} \tag{37}$$

for $i = 1, \dots, I$, $j = 1, \dots, J$ and $n = N, N - 1, \dots, 0$, where we define $P_{i,j}^{n+1/2} = P_{i,j}^{n+1}$ in the semi-implicit scheme and $P_{i,j}^{n+1/2} = P_{i,j}^n$ in the fully implicit method. As for the explicit scheme, the the final condition and boundary conditions are defined in equations (25)-(29).

Some simple algebraic manipulations show that this scheme can be written on the form

$$\begin{aligned} & e_{i,j}\, P^n_{i-1,j-1} + [b_j - e_{i,j}]\, P^n_{i,j-1} + [a_i - e_{i,j}]\, P^n_{i-1,j} \\ & -[1 + 2a_i + 2b_j - 2e_{i,j} + c_i + d_j + r\Delta t]\, P^n_{i,j} \\ & +[a_i - e_{i,j} + c_i]\, P^n_{i+1,j} + [b_j - e_{i,j} + d_j]\, P^n_{i,j+1} + e_{i,j}\, P^n_{i+1,j+1} \\ & = -P^{n+1}_{i,j} - \frac{\epsilon C \Delta t}{P^{n+1/2}_{i,j} + \epsilon - q_{i,j}}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} & a_i = a(x_i), \quad b_j = b(y_j), \quad c_i = c(x_i), \\ & d_j = d(y_j), \quad e_{i,j} = e(x_i, y_j). \end{aligned}$$

Note that the semi-implicit scheme, $P^{n+1/2}_{i,j} = P^{n+1}_{i,j}$ gives a system of linear algebraic equations, whereas the fully implicit scheme, $P^{n+1/2}_{i,j} = P^n_{i,j}$ leads to a system of non-linear equations.

4 Analysis in the case of independent assets

In this section we will prove that our schemes satisfy the early exercise constraint. Our analysis will only cover the case of independent assets, i.e. we will assume the

$$\rho = 0$$

throughout this section. Unfortunately we have not been able to derive similar results in the correlated case. However, such problems will be addressed by numerical experiments in Section 5.

4.1 Analysis of the explicit scheme

Theorem 1 *Assume that $\rho = 0$ and that $C \geq rE$. Then the approximate option values generated by the explicit scheme (36) satisfy*

$$P^n_{i,j} \geq \max(q(x_i, y_j), 0), \quad i = 0, \ldots, I+1,\ j = 0, \ldots, J+1 \tag{39}$$

and $n = N+1, N, \ldots, 0$, provided that

$$\Delta t \leq \frac{h^2}{\sigma_1 x_\infty^2 + \sigma_2 y_\infty^2 + (r - D_1) h x_\infty + (r - D_2) h y_\infty + r h^2 + \frac{C}{\epsilon} h^2}. \tag{40}$$

Proof. In the case of independent assets the function F, defined in equation (36), takes the form

$$\begin{aligned} F(V_1, V_2, V_3, V_4, V_5, V_6, V_7, q, x, y) = \\ & b(y)\, V_2 + a(x)\, V_3 \\ & +[1 - 2a(x) - 2b(y) - c(x) - d(y) - r\Delta t]\, V_4 \\ & +[a(x) + c(x)]\, V_5 + [b(y) + d(y)]\, V_6 \\ & + \frac{\epsilon C \Delta t}{V_4 + \epsilon - q}, \end{aligned} \tag{41}$$

i.e. $e(x, y) = 0$ for all $x, y \geq 0$, see (35). Clearly, for all $x, y \geq 0$

$$\frac{\partial F}{\partial V_2}, \frac{\partial F}{\partial V_3}, \frac{\partial F}{\partial V_5}, \frac{\partial F}{\partial V_6} \geq 0, \tag{42}$$

and for $V_4 \geq q$

$$\frac{\partial F}{\partial V_4} \geq 0, \tag{43}$$

provided that Δt satisfies (40).

Assume that the inequality (39) holds at time step t_n. From the definition (36) of our scheme, and inequalities (42) and (43), we find that

$$\begin{aligned} P_{i,j}^{n-1} &= F(0, P_{i,j-1}^n, P_{i-1,j}^n, P_{i,j}^n, P_{i+1,j}^n, P_{i,j+1}^n, 0, q_{i,j}, x_i, y_j) \\ &\geq F(0, q_{i,j-1}, q_{i-1,j}, q_{i,j}, q_{i+1,j}, q_{i,j+1}, 0, q_{i,j}, x_i, y_j). \end{aligned} \tag{44}$$

Recall the definition (18) of the payoff function q at time $t = T$ of the basket option. Thus,

$$\begin{aligned} q_{i,j-1} = q_{i,j} + \alpha_2 h, \quad q_{i-1,j} = q_{i,j} + \alpha_1 h, \\ q_{i+1,j} = q_{i,j} - \alpha_1 h, \quad q_{i,j+1} = q_{i,j} - \alpha_2 h, \end{aligned}$$

and consequently

$$\begin{aligned} P_{i,j}^{n-1} &\geq b_j \alpha_2 h + a_i \alpha_1 h + q_{i,j} - r\Delta t q_{i,j} - [a_i + c_i]\alpha_1 h - [b_j + d_j]\alpha_2 h \\ &\quad + \frac{\epsilon C \Delta t}{q_{i,j} + \epsilon - q_{i,j}} \\ &= q_{i,j} - r\Delta t q_{i,j} - (r - D_1)\frac{\Delta t}{h} x_i \alpha_1 h - (r - D_2)\frac{\Delta t}{h} y_j \alpha_2 h + C\Delta t \\ &= q_{i,j} - r\Delta t q_{i,j} - r\Delta t(x_i \alpha_1 + y_j \alpha_2) + D_1 \Delta t x_i \alpha_1 + D_2 \Delta t y_j \alpha_2 \\ &\quad + C\Delta t \\ &\geq q_{i,j} - r\Delta t q_{i,j} - r\Delta t(E - q_{i,j}) + C\Delta t, \end{aligned}$$

where we have used the definition (18) of q. Therefore, if $C \geq rE$ then

$$P_{i,j}^{n-1} \geq q_{i,j} + (C - rE)\Delta t \geq q_{i,j}.$$

Furthermore, from equations (41) and (44) and the assumption that $P_{i,j}^n$ satisfies (39), i.e. $P_{i,j}^n \geq 0$ and $P_{i,j}^n \geq q_{i,j}$, we find that

$$P_{i,j}^{n-1} \geq 0,$$

and, hence the desired result follows by induction. ■

4.2 Analysis of the semi-implicit and fully implicit schemes

Theorem 2 *For every $C \geq rE$ the the approximate option prices $\{P_{i,j}^n\}$ defined by the fully implicit scheme (37) satisfy*

$$P_{i,j}^n \geq \max(q(x_i, y_j), 0), \quad i = 0, \dots, I+1,\ j = 0, \dots, J+1, \tag{45}$$

and $n = N+1, N, \dots, 0$. Similarly, if $C \geq rE$, and in addition

$$\Delta t \leq \frac{\epsilon}{rE}, \tag{46}$$

the numerical option prices generated by the semi-implicit version of (37) satisfy the lower bound (45).

Proof. In a straight forward manner it follows that the difference

$$u_{i,j}^n = P_{i,j}^n - q_{i,j}$$

between the approximate option value $P_{i,j}^n$ and q, used in the payoff function at expiry (18) and (23), satisfies the equation

$$\begin{aligned} &[1 + 2a_i + 2b_j + c_i + d_j + r\Delta t]\, u_{i,j}^n = \\ &u_{i,j}^{n+1} + b_j\, u_{i,j-1}^n + a_i\, u_{i-1,j}^n + [a_i + c_i]\, u_{i+1,j}^n \\ &+[b_j + d_j]\, u_{i,j+1}^n + \frac{\epsilon C \Delta t}{u_{i,j}^{n+1/2} + \epsilon - q_{i,j}} - r\Delta t E, \end{aligned}$$

cf. equation (38) (and recall that $\rho = 0$, i.e. $e(x, y) = 0$ for all $x, y \geq= 0$). Next, by defining

$$u^n = \min_{i,j} u_{i,j}^n$$

it follows that

$$\begin{aligned} &[1 + 2a_i + 2b_j + c_i + d_j + r\Delta t]\, u^n \geq \\ &u_{k,l}^{n+1} + b_j\, u^n + a_i\, u^n + [a_i + c_i]\, u^n \\ &+[b_j + d_j]\, u^n + \frac{\epsilon C \Delta t}{u_{k,l}^{n+1/2} + \epsilon - q_{i,j}} - r\Delta t E, \end{aligned}$$

where k and l are indices such that $u^n_{k,l} = u^n$. Hence, we conclude that

$$[1 + r\Delta t]\, u^n \geq u^{n+1}_{k,l} + \frac{\epsilon C \Delta t}{u^{n+1/2}_{k,l} + \epsilon - q_{i,j}} - r\Delta t E. \tag{47}$$

Having established inequality (47) the result follows exactly as for the single-asset option problem analysed in [11]. The rest of the proof is therefore omitted.

■

5 Numerical experiments

In the derivation and analysis of the schemes above we only assumed that the boundary conditions fulfilled the constraint (4), imposed by the early exercise feature of the contract. Clearly, in order to perform numerical experiments we need to fully specify these boundary conditions. Since we are considering put options the contract gets worthless as the price of either of the assets tend to infinity, i.e.

$$\begin{aligned} G_1(y,t) &= 0, \quad y \geq 0,\ t \in [0,T], \\ G_2(x,t) &= 0, \quad x \geq 0,\ t \in [0,T], \end{aligned}$$

see equations (16) and (17). Next, it follows from the lognormal distribution model of the assets, cf. e.g. [14], that if one of the assets is zero at time t^* then the asset will be worthless at any time $t \geq t^*$. Hence, it follows that g_1 and g_2, in equations (14) and (15), are the solutions of the associated single-asset American put problems,

$$\frac{\partial g_1}{\partial t} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 g_1}{\partial x^2} + (r - D_1)x\frac{\partial g_1}{\partial x} - r g_1 = 0 \quad \text{for } x > \bar{x}(t) \text{ and } 0 \leq t < T, \tag{48}$$

$$g_1(x,T) = \max(E - \alpha_1 x, 0) \quad \text{for } x \geq 0, \tag{49}$$

$$\frac{\partial g_1}{\partial x}(\bar{x}(t), t) = -\alpha_1, \tag{50}$$

$$g_1(\bar{x}(t), t) = E - \alpha_1 \bar{x}(t), \tag{51}$$

$$\lim_{x\to\infty} g_1(x,t) = 0, \tag{52}$$

$$\bar{x}(T) = E/\alpha_1, \tag{53}$$

$$g_1(x,t) = E - \alpha_1 x \quad \text{for } 0 \leq x < \bar{x}(t), \tag{54}$$

and a similar problem for g_2. Here $\bar{x}(t)$ represents the free (and moving) boundary, see e.g. [7], [10] or [14].

In all the experiments below we will apply the penalty method, derived for single-assets problems in [11], to compute an approximate solution of

(48)-(54), i.e. to compute $(g_1)_i^n$ and $(g_2)_j^n$ in equations (26) and (27).

The following model parameters are used throughout this section,

$$
\begin{aligned}
r &= 0.1, & & \\
\sigma_1 &= 0.2, & \sigma_2 &= 0.3, \\
\alpha_1 &= 0.6, & \alpha_2 &= 0.4, \\
D_1 &= 0.05, & D_2 &= 0.01, \\
E &= 1.0, & & \\
T &= 1.0. & &
\end{aligned}
$$

The correlation parameter ρ is identical to zero in the independent case, and $\rho = 0.25$, 0.5 and 0.75 in the correlated cases. In order to perform simulations, we must choose an upper limit for the solution domain, that is a domain where option values outside are regarded worthless. For our set of model parameters, we have used $x_\infty = y_\infty = 4$.

The implemetation of the finite difference schemes is done whithin the Diffpack[2] framework.

Numerical results for the fully implicit scheme are not provided, based on the lack of efficiency of the non-linear scheme examined in [11].

In order to illustrate the properties stated in Theorem 1 and Theorem 2, we compute the difference between the numerical solutions and the early exercise contraint, i.e. we compute

$$\phi = \min_{i,j,n}(P_{i,j}^n - \max(q_{i,j}, 0)), \tag{55}$$

for different values of ϵ.

5.1 Independent assets

We first compare the explicit and semi-implicit schemes with respect to effieciency, i.e. we compare the CPU time for given spatial resolutions, choosing time step sizes according to (40) for the explicit scheme and (46) for the semi-implicit scheme.

The linear system of algebraic equations in the semi-implicit case is solved with a stable bi-conjugate gradient method, called Bi-CGSTAB c.f. [12], using the modified incomplete LU factorization, MILU, as preconditioner[3]. We have used a relative residual convergence criterion for the iterative solver, i.e. the iteration process was stopped when $||r_k||/||r_0|| \leq 10^{-4}$, where r_k represents the residual vector at iteration k.

[2]See [6] for further information regarding the Diffpack library.

[3]For analytical and numerical studies of preconditioned iterative solvers confer e.g. [4] and [9].

Table 1: CPU time comparision of the explicit and semi-implicit schemes. We choose Δt to satisfy the bounds given in (40) and (46). We have uniform mesh size, $N = I \cdot I$, where I is the number of nodes in each space direction. In all experiments, $\phi = 0$, c.f. equation (55).

			Explicit		Semi-implicit	
h	N	ϵ	CPU-time	Δt	CPU-time	Δt
0.1	1 681	0.01	$1.4s$	$4.5 \cdot 10^{-3}$	$1.3s$	0.1
0.05	6 561	0.01	$20.4s$	$1.2 \cdot 10^{-3}$	$5.3s$	0.1
0.01	160 901	0.01	$1.28 \cdot 10^4 s$	$4.8 \cdot 10^{-5}$	$157.4s$	0.1
0.1	1 681	0.001	$2.0s$	$3.2 \cdot 10^{-3}$	$11.2s$	0.01
0.05	6 561	0.001	$22.5s$	$1.1 \cdot 10^{-3}$	$45.4s$	0.01
0.01	160 901	0.001	$1.29 \cdot 10^4 s$	$4.8 \cdot 10^{-5}$	$1.2 \cdot 10^3 s$	0.01

The result are given in Table 1. We observe that the severe restrictions on the time step size in the explicit case makes this scheme slow for fine grained meshes. On the other hand, we experience fast convergance of the preconditioned iterative solver used in the semi-implicit case. Typically, the BICGSTAB iteration converges in 5 iterations. Together with the mild restriction on the time step size, the latter method is the most attractive as the mesh is refined.

In Section 4, we showed that when certain conditions on the time step size and penalty function are satisfied, the early exercise constraint is fulfilled in a discrete sense. We want to test the sharpness of the properties expressed in Theorem 1 and Theorem 2 by violating these restrictions, looking for negative values of ϕ.

We start by increasing the time step size by 15% for the explicit scheme. When $\epsilon = 0.01$, $\phi = -4.8 \cdot 10^{49}$ which clearly violates (39).

We also break the milder restriction for the semi-implicit scheme by choosing $\Delta t = 10^{-2}$ and $\epsilon = 10^{-4}$. Again, we experience negative values of ϕ, i.e. $\phi = -9.7 \cdot 10^{-2}$.

Finally we subtract 10% from the constant C in the penalty term, i.e. we choose $C = 0.9 \cdot rE$, in the semi-implicit case. We use $\epsilon = 10^{-2}$, $\Delta t = 10^{-2}$ and $h = 0.1$ in this experiment. Now the penlaty term is weaker, exerting less force on the solution as it approaches the barrier. We obtain $\phi = -6.7 \cdot 10^{-4}$, thus we are not able to keep the solution on the proper state space.

5.2 Correlated assets

The results in Section 4 are obtained by assuming that the underlying assets are independent. We provide a range of numerical experiments indicating that the early exercise constraint is fulfilled in the case of correlated assets as well. We choose different values for the correlation parameter ρ between

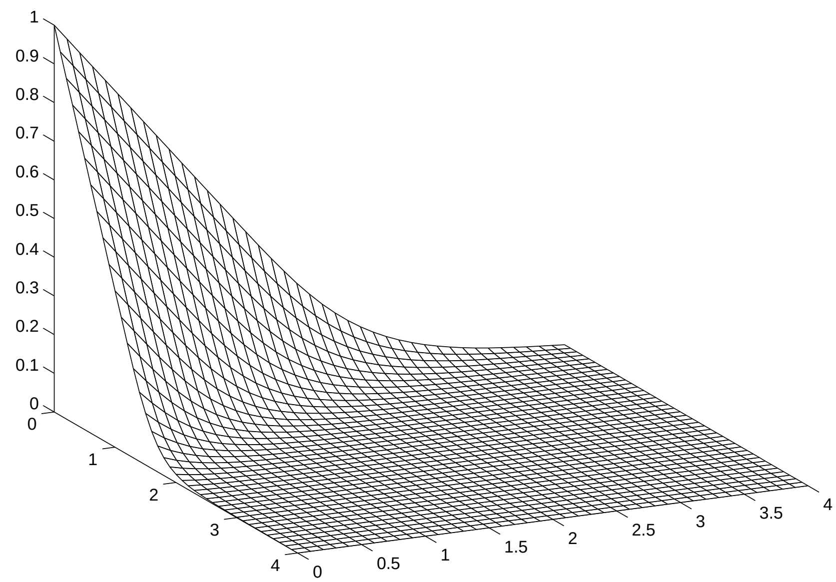

Figure 1: A plot of the solution obtained by the semi-implicit scheme, at time $t = 0$ of the two-factor model problem with correlation $\rho = 0.5$. We have used $\epsilon = 10^{-2}$, $x_\infty = y_\infty = 4.0$ and $h = \Delta t = 10^{-1}$.

assets S_1 and S_2, $\rho = 0.25$, 0.5 and 0.75. The experiments given in Table 1 has been run with the new correlation parameter settings, and in all cases $\phi = 0$, thus the early exercise constraint is fullfilled for both schemes.

A plot of the numerical solution computed by the semi-implicit scheme at time $t = 0$ for $\rho = 0.5$ is given in Figure 1.

Remark

In the case of correlated underlying assets, we can construct a final condition that satisfies the early exercise contraint, but leads to a solution violating this constraint at the first time step. To see this we consider (34), and let $\sigma_1 = \sigma_2 = \sigma$, $E = 1$, $\alpha_1 = \alpha_2 = 1/2$, $\rho = 1/2$ and $V_1 = V_2 = V_4 = V_5 = V_6 = V_7 = 0$. We consider a mesh point far away from the barrier, i.e. $x = 2E$ and $y = 3x$. Then the function F, defined in equation (34) will be negative provided that

$$V_3 > \frac{h^2 \epsilon C}{\sigma^2(\epsilon + 3)}. \tag{56}$$

However, for the final conditions on the form (9), which makes financial sense, this behaviour was not experienced in our numerical experiments.

For other contracts, this effect might be seen. Further investigations should be carried out in order to provide further insight into this matter.

6 Conclusion

We have presented a penalty method for solving multi-asset American put option problems. An explicit, semi-implicit and a fully implicit finite difference scheme utilising a penalty term have been derived. For independent underlying assets, conditions on the discretization parameters and the penalty term have been established that assure that the numerical solution satisfies the constraint arising from the early exercise feature of the contract.

We have run several numerical experiments for the explicit and semi-implicit schemes. We prefer the semi-implicit scheme to the explicit for fine grained meshes due to the computational efficiency of the semi-implicit scheme. Experiments indicate that the constraints derived in Section 4 are sharp. In the case of correlated underlying assets we have not achieved similar theoretical results. However, experiments indicate that for our model parameters, the solution of the explicit and semi-implicit schemes satisfy the early exercise constraint. We finally present an example final condition that leads to violation of the early exercise constraint for the explicit scheme in the correlated case.

References

[1] F. Zapatero A. Ibáñez. Monte Carlo valuation of american options through computation of the optimal exercise frontier. Technical report, Finance and Business Economics Department. Marshall School of Business. USC Los Angeles, 1999.

[2] F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *J. Political Economics*, 81:637–659, 1973.

[3] P.P. Boyle, J. Evnine, and S. Gibbs. Numerical Evaluation of Multivariate Contingent Claims. *Review of Financial Studies*, 2(2):241–250, 1989.

[4] A. M. Bruaset. *A survey of preconditioned iterative methods.* Longman Scientific & Technical, 1995.

[5] N. Clarke and A. K. Parrot. The multigrid solution of two-factor American put options. Technical report, Oxford Computing Laboratory, 1996.

[6] Diffpack. See http://www.nobjects.no/ for further information.

[7] D. Duffie. *Dynamic asset pricing theory.* Princeton University Press, 1996.

[8] P. A. Forsyth, K. R. Vetzal, and R. Zvan. A Finite Element Approach to the Pricing of Discrete Lookbacks With Stochastic Volatility. *Applied Mathematical Finance*, 6:87–106, 1999.

[9] W. Joubert. Iterative methods for the solution of nonsymmetric systems of linear equations. Technical Report CNA-242, Center for Numerical Analysis, University of Texas at Austin, 1990.

[10] Y. K. Kwok. *Mathematical models of financial derivatives.* Springer-Verlag, 1998.

[11] B. F. Nielsen, O. Skavhaug, and A. Tveito. Penalty and front-fixing methods for the numerical solution of American option problems. Preprint 2000-5, Department of Informatics, University of Oslo, 2000.

[12] H.A. van der Vorst. Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, 13(2):631–644, March 1992.

[13] P. Wilmott. *Derivatives, The theory and practice of financial engineering.* John Wiley & Sons, 1998.

[14] P. Wilmott, J. Dewynne, and S. Howison. *Option Pricing, Mathematical models and computation.* Oxford Financial Press, 1993.

[15] Y. I. Zhu and Y. Sun. The singularity-separating method for two-factor convertible bonds. *The Journal of Computational Finance*, 3(1):91–110, 1999.

[16] R. Zvan, P. A. Forsyth, and K. R. Vetzal. Penalty Methods for American Options With Stochastic Volatility. *Journal of Computational and Applied Mathematics*, 91(2):199–218, 1998.

[17] R. Zvan, P.A. Forsyth, and K.R. Vetzal. A Finite Volume Approach For Contingent Claims Valuation. Technical report, University of Waterloo, Waterloo, Ontario, 1999.