Measurement Error Models for the Norwegian Minke Whale Surveys 1996-2001

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Abstract
In the Norwegian minke whale surveys, the position of an observed whale relative to the vessel is given by the radial distance and the angle from the transect line. In the present paper, the bias and the variability in the observed radial distances and angles is estimated from buoy sighting experiments. The variability of observed angles is further modified by using data for duplicate observations in real sightings.

1 Introduction

To obtain abundance estimates of minke whales in the northeastern Atlantic, surveys have been conducted in each year in the period 1996-2001 (Skaug et. al. 2002). As a part of the analysis, there is a need to quantify the bias and variability in the relative positions of sighted surfacing. The present paper presents measurement error models for the relative positions, based on a parallel set of buoy experiments in the years 1996-2001, and on duplicate observations from the real sightings in the same period. The paper updates the results of Schweder (1997), based on new buoy data and modified models.

In a real survey, the position of a surfacing whale is recorded by an observer as the estimated radial distance from the observer to the spot of water where the whale was seen and the angle between the sighting line and the transect line.

This situation is imitated in buoy sighting experiments where observed radial distances and angles to buoys are estimated from the platform, and at same time measured by radar.

From the buoy data we have estimated a model for the observed radial distance as a function of the true radial distance, where both the bias and the variability depends on the true distance. Similarly, we have modelled the observed angle as a function of the true angle, where the bias and the variability depends on the true angle. The variability of observed angles is further modified by using data for duplicate observations in real sightings. These models are slightly more complex than those suggested by Schweder (1997).

2 Data

The buoy sighting experiments were conducted the following way: Two buoys with radar reflectors are dropped into the sea at a distance 1000-2000 meters from the vessel, one on the port and the other on the starboard side. The vessel moves towards the buoys at cruising speeds (about 10 knots). The vessels have two independent observers located at platforms A and B respectively. At a signal, they are asked to estimate the position of one of the two buoys, given as the radial distance and the angle from the vessel direction. The radial distance is estimated by eye, whereas the angle is measured by an angle board. At the same time the true radial distance and angle are measured by radar and recorded. This procedure is repeated when the vessel moves towards the buoys, but with a random switching between the buoys. Since several recordings are made within each new drop of buoys, the data will be positively serially correlated. This type of experiments were called experiment type B in Schweder (1997).

Such experiments were conducted each years in the period 1996-2001, for six different vessels, and for various observers.

We will let R and r denote the observed and true radial distances, respectively, measured in metres. Furthermore, Θ and θ denotes the observed and true angles, measured in degrees, respectively. Angles on starboard are positive, and angles on port are negative.

A few times, the observer have obviously looked at the wrong buoy. Such observations are deleted. One example is when the true angle is 31 degrees, one observer has recorded 34 degrees, but the other has recorded -34 degrees. One observation with abs(θ) > 90 degrees is deleted as well. Furthermore, for each platform we use only the observations where all four corresponding measurements of R, r, Θ, and θ were recorded.

The resulting data set consists of 1988 observations with corresponding values of R, r, Θ, and θ. This data set is used for the estimation of the models for the radial distances and the angles. For most of these observations, the buoy were observed from both platforms. When these are grouped in pairs, we have 820 observations with observed radial distances (R_A, R_B) and angles (Θ_A, Θ_B), where the subscripts denotes platform A or B.

In addition to the data from the buoy experiments, we have 109 pairs of duplicate observations from the real sightings in the 1996-2001 surveys. These data are discussed in Section 6.

3 The radial distance model

A reasonable starting point is to assume that R/r is log normally distributed, or equivalent, that $\log(R)$-$\log(r)$ is normally distributed. This is confirmed by Figure 1, which shows $\log(R)$ and $\log(R)$-$\log(r)$ versus $\log(r)$.

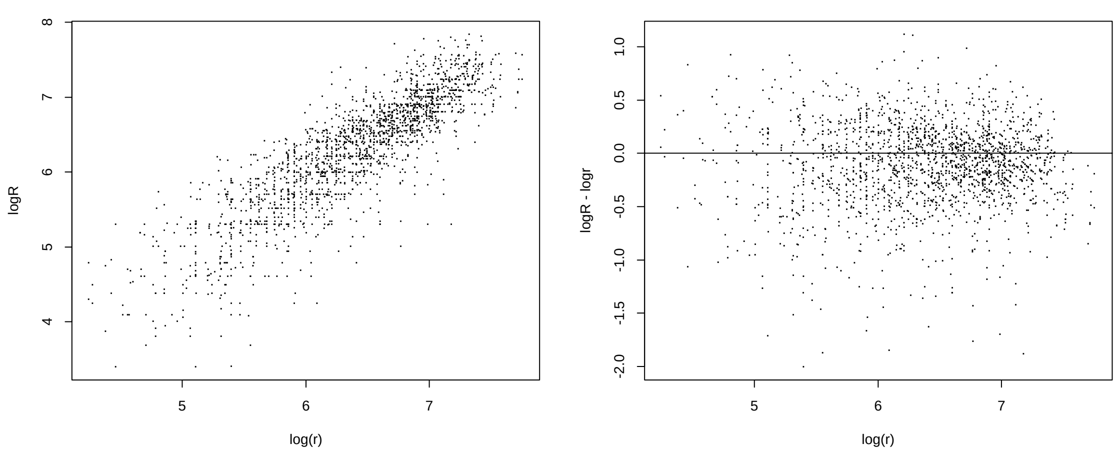

Figure 1 Plot of log(*R*)-log(*r*) versus log(*r*).

We estimate the model

$$\log(R) = \beta_0 + \beta_1 \log(r) + \varepsilon \quad , \quad \varepsilon \sim N(0, \sigma^2) \quad , \tag{1}$$

by maximum likelihood. From Figure 1 we will expect that β_0 is close to 0 and that β_1 is close to 1. The estimates and their standard errors (for the β's only) are shown in the "estimate 1" column of Table 1. The standard errors in Table 1 ignores the positive serial correlation in the data, and are biased downwards. From a simple bootstrap exercise, Schweder (1997) found that the true standard deviations of the parameter estimators could be four times as high as the nominal standard errors. With this in mind, β_1 is clearly not significant different from 1. We therefore also fit a model with β_1 fixed to 1 (the "estimate 2" column of Table 1.

Table 1 Estimates of model (1), with nominal standard errors from standard linear regression in parenthesis. Estimate 1: Both parameters estimated. Estimate 2: β_1 set to 1.

parameter	estimate 1	estimate 2
β_0	-0.191 (0.085)	-0.110 (0.009)
β_1	1.013 (0.013)	1.000
σ	0.388	0.388

We have also estimated a general additive model (Hastie and Tibshirani, 1990) where the linear term in (1) has been replaced by a spline function with approximately four degrees of freedom. This analysis gave no evidence for a non-linear relationship between log(*R*) and log(*r*).

The error in log(*R*) may potentially depend on θ as well. If so, the dependency has to be of abs(θ), i.e. symmetric around the transect line. When adding a spline function of abs(θ) with four degrees of freedom to model (1), log(*R*) seems to be slightly linearly dependent on abs(θ). Therefore, abs(θ) is added linearly to model (1). The regression coefficient is estimated to 0.0016, with a standard error of 0.0005. Having the downward bias of the standard error in mind, the angle effect is hardly significant. Anyway, the resulting estimate of σ is only 0.001 less than in model (1). From this, no angle effect is included in the model for radial distance.

We now study the residuals $\hat{\varepsilon} = \log(R) - (\hat{\beta}_0 + \hat{\beta}_1 \log(r))$ from model (1). The left panel of Figure 2 shows the residuals versus log(*r*). The standard deviation σ is seen to decrease for increasing value of log(*r*). To investigate this further, we assume for the moment that

$$\hat{\varepsilon}^2 \sim \mathrm{Gamma}(\sigma^2(r), \zeta) \quad , \qquad (2)$$

where $\sigma^2(r) = E(\hat{\varepsilon}^2)$ is the expectation in the Gamma distribution and ζ is the shape parameter. Further, we assume the following structure

$$\log(\sigma^2(r)) = 2\log(\sigma(r)) = s(\log(r)) \quad , \tag{3}$$

where s is a spline function with four degrees of freedom. We fit this model by the framework of generalized additive models. The estimated spline function (centred around 0) with corresponding 95% confidence interval are shown in the right panel of Figure 2. The plot clearly indicates a linear relationship. The non-linearities in the ends are not significant.

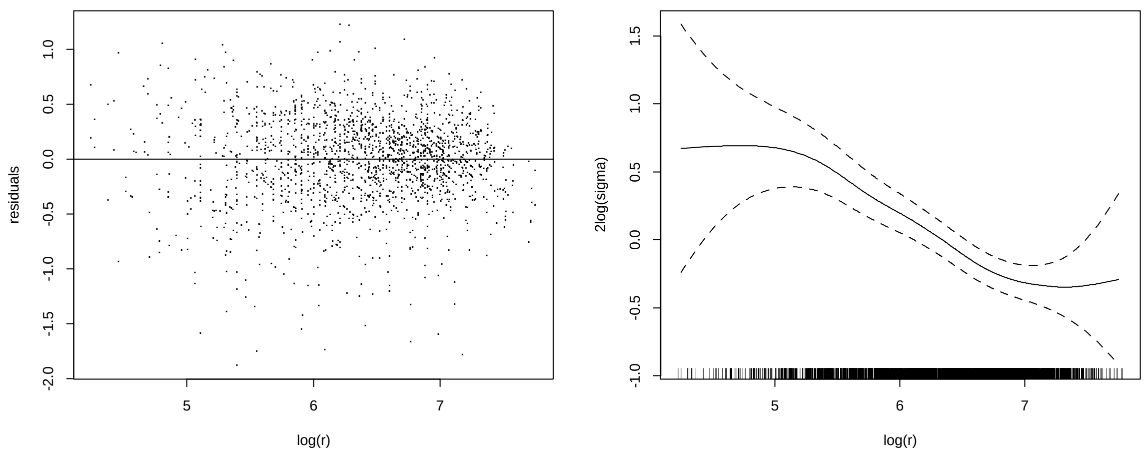

Figure 2 Left panel: Residuals from model (1) versus log(*r*).
Right panel: The spline function estimate of $\log(\sigma^2(r))$ versus log(r), with 95% confidence limits (ignoring serial correlation). The black tick marks at the bottom show the distribution of log(*r*).

The above analysis leads to the model

$$\begin{aligned} \log(R) &= \beta_0 + \beta_1 \log(r) + \varepsilon(r) \quad , \\ \varepsilon(r) &\sim N(0, \sigma^2(r)) \quad , \\ \frac{1}{2}\log(\sigma^2(r)) &= \log(\sigma(r)) = \alpha_0 + \alpha_1 \log(r) \quad . \end{aligned} \tag{4}$$

The four parameters are simultaneously estimated by maximum likelihood (Table 2). The estimate of β_1 is very close to 1, and we refit the model with β_1 fixed to 1 (Table 2).

Table 2 Estimates of model (4). Estimate 1: All parameters estimated. Estimate 2: β_1 set to 1.

parameter	estimate 1	estimate 2
β_0	-0.120	-0.108
β_1	1.002	1.000
α_0	0.539	0.542
α_1	-0.237	-0.237

Finally, we want to check the normality assumption of model (4). We calculate standardized residuals from the last model (with $\hat{\beta}_1 = 1$) as $[\log(R) - (\hat{\beta}_0 + \log(r))]/\hat{\sigma}(r)$, which should have standard deviation around 1. Figure 3 shows that the standardized residuals have a slightly skewness to the left, but we find this acceptable. We therefore conclude that the model (4) is a satisfactory model for the radial distance, with parameters given in the right most column of Table 2.

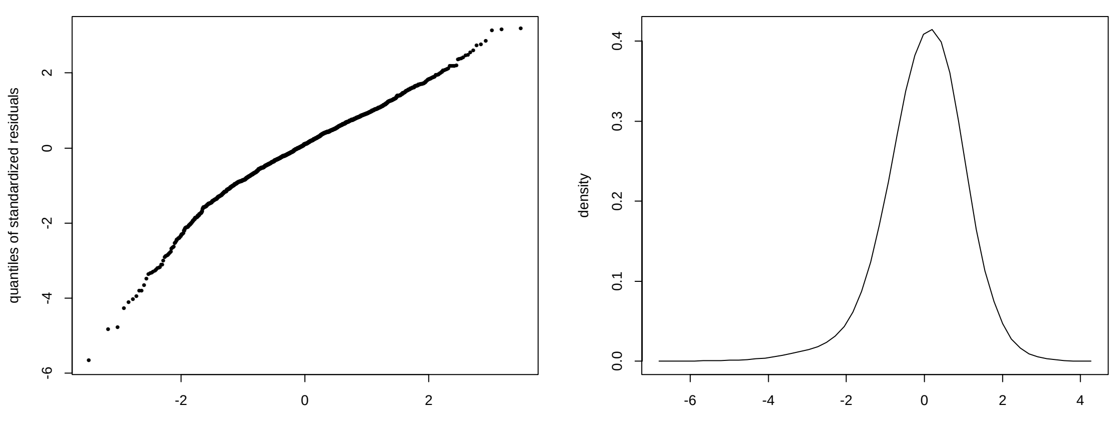

Figure 3 Left panel: Normal score plot for standardized residuals from model (4). Right panel: Smoothed density plot of the standardized residuals.

Inserting the parameter estimates into model (4), and transforming back to the original scale, gives our proposed measurement error model for the radial distances, based on the buoy data:

$$R = \exp(-0.108) \cdot r \cdot \exp(\varepsilon(r)) = 0.898 \cdot r \cdot \exp(\varepsilon(r)) \quad ,$$
$$\varepsilon(r) \sim N(0, \sigma^2(r)) \quad , \qquad (5)$$
$$\sigma(r) = \exp(0.542 - 0.237\log(r)) = 1.71 \cdot r^{-0.237} \quad .$$

The standard deviation and bias of this model is shown in Table 3.

Table 3 Standard deviation and bias of model (5).

r	100	500	1000	2000
$\sigma(r)$	0.58	0.39	0.33	0.28
$E(R/r)$	1.06	0.97	0.95	0.93

4 The angle model

Figure 4 shows Θ and the difference Θ-θ plotted against θ. (The white stripes that can be seen in the right panel are only due to Θ and θ being recorded in whole degrees). If Θ were an unbiased measure of θ, Θ-θ should vary around 0 for all values of θ. However, there seems to be an increasing linear relationship: when the buoy is on the starboard side (θ is positive), the observer's estimate Θ tends to be too large, i.e. too far to starboard, and vice versa when the buoy is on the port side. When the buoy is close to the vessel direction ($\theta \approx 0$), there seems to be no bias, which is reasonable.

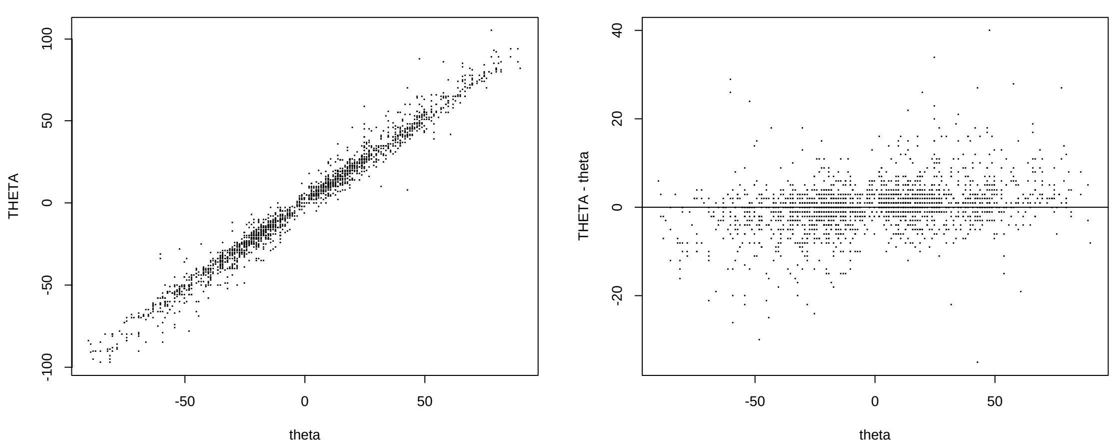

Figure 4 Plot of Θ and Θ-θ versus θ.

We therefore start by fitting the model

$$\Theta = \beta\theta + \varepsilon \qquad , \qquad \varepsilon \sim N(0, \sigma^2) \quad . \qquad (6)$$

again using ordinary least squares. The model contains no constant term, because when θ=0, $E(\Theta)$ should obviously be 0. The estimates are shown in Table 4. We conclude that β is significant different from 1, even if we take into account that the estimated standard error is biased downwards by a factor of 4, cf. the corresponding analysis of radial distances. The estimated value of σ is 5.41, which can be compared to the standard deviation of Θ-θ, which is 5.72. Thus, even if the bias is clear, it is relatively small compared to the additional random error.

Table 4 Estimates of model (6) with standard error disregarding serial correlation.

parameter	estimate
β	1.054 (0.004)
σ	5.41

In model (6), we ignored the potential dependency on the true distance *r*. One explanation for the patterns in Figure 4 may be the movement of the vessel. When the observer gets the signal to observe the buoy (or in a real sighting; when the observer sees the whale), it will take some time to measure the angle. Since the vessel moves forwards, the measured angles will tend to be too far out to each of the sides due to the delay in the angle measurement. To investigate this further, we calculated the angle velocity $x = d\theta / dl$, i.e. the alteration dθ in angle when the boat moves a short distance dl, which depends on both θ and *r*, and fitted the model $\Theta = \theta + \beta x + \varepsilon$. This model gave a worse fit than model (6). We also fitted a linear model with θ, log(*r*) and the cross product $\theta \cdot \log(r)$ as explanatory variables. Even if this model improved slightly on model (6), with $\hat{\sigma} = 5.34$, we have chosen to keep model (6) because of it’s simplicity.

The left panel in Figure 5 shows the residuals $\hat{\varepsilon} = \Theta - \hat{\beta}\theta$ from model (6). There is a tendency that the variance increases when abs(θ) increases from 0. In analogy to the analysis of model (1), we assume for the moment that the residuals follows a model like that specified by (2) and (3), except that σ depends on θ instead of *r*. The estimated spline function is shown in the right panel of Figure 5. The logarithm of σ^2 seem to increase linearly when abs(θ) increase from 0 degrees to around 55 degrees. The estimated spline function decreases when θ increase further, but this is insignificant (again taking into account that the uncertainty is underestimated).

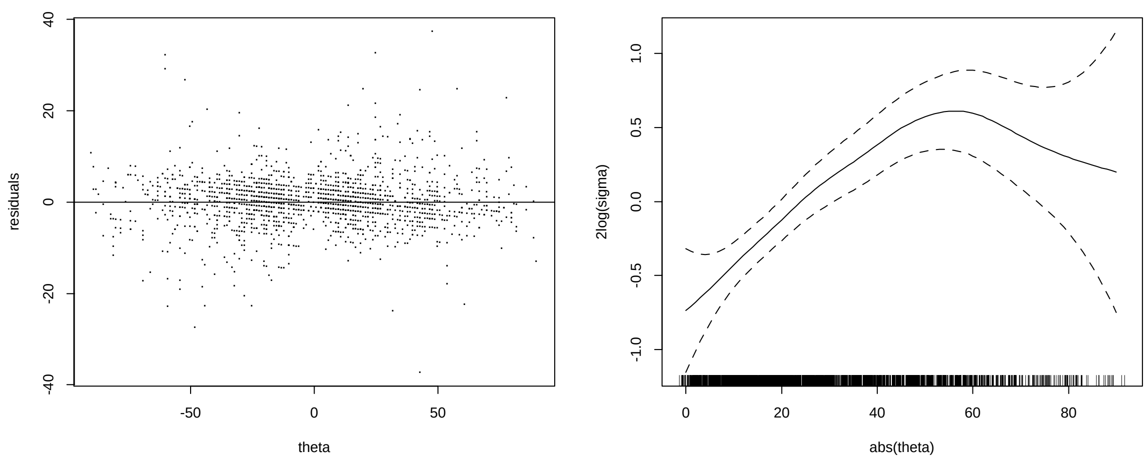

Figure 5 Left panel: Residuals from model (6) versus θ.
Right panel: The spline function s(abs(θ)) versus abs(θ), with 95% confidence limits.

We therefore specify the following model where the logarithm of σ^2 increase linearly by θ up to 55 degrees, and remains constant for higher values of θ:

$$\begin{aligned} \Theta &= \beta\theta + \varepsilon(\theta) \quad , \\ \varepsilon(\theta) &\sim N(0, \sigma^2(\theta)) \quad , \\ \frac{1}{2}\log(\sigma^2(\theta)) &= \log(\sigma(\theta)) = \alpha_0 + \alpha_1 \min(\mathrm{abs}(\theta), 55) \quad . \end{aligned} \tag{7}$$

The three parameters are estimated simultaneously by maximum likelihood, and shown in the “estimate normal” column of Table 5.

Table 5 Estimates of the model (7). Estimate normal: Assuming normal distribution. Estimate t: Assuming t distribution.

parameter	estimate normal	estimate t
β	1.057	1.045
α_0	1.336	1.312
α_1	0.0117	0.00918
ν	-	5
α_0'	-	1.057

To check the normality assumption of model (7), we calculate the standardized residuals as $(\Theta - \hat{\beta}\theta)/\hat{\sigma}(\theta)$. Figure 6 shows that the standardized residuals are symmetric, but with heavy tails.

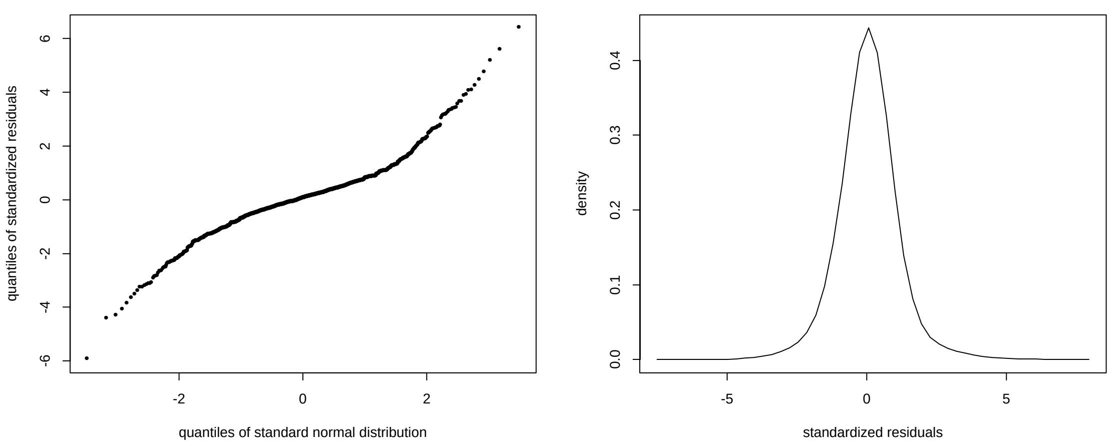

Figure 6 Left panel: Normal score plot for standardized residuals from model (7). Right panel: A smoothed density plot of the standardized residuals.

We therefore alter the normality assumption in (7) to

$$\varepsilon(\theta) = s(\theta) \cdot t_\nu \quad , \tag{8}$$

where t_ν is t distributed with ν degrees of freedom, and $s(\theta)$ is defined such that the last line in (7) is still valid. We assume that $\nu \geq 3$, because this is necessary for the variance of t_ν to exist, which then is $\mathrm{Var}(t_\nu) = \nu/(\nu-2)$. Since $\sigma^2(\theta) = \mathrm{Var}(\varepsilon(\theta)) = s^2(\theta) \cdot \mathrm{Var}(t_\nu) = s^2(\theta) \cdot \nu/(\nu-2)$, $s(\theta)$ is given by

$$\frac{1}{2}\log(s^2(\theta)) = \log(s(\theta)) = \alpha_0' + \alpha_1 \min(\mathrm{abs}(\theta), 55) \quad ,$$
$$\alpha_0' = \alpha_0 - \frac{1}{2}\log(\nu/(\nu-2)) \quad . \tag{9}$$

The parameters, including ν, are first estimated simultaneously by maximum likelihood, giving $\hat{\nu}^{ML} = 3$. However, a plot of the quantiles of the standardized residuals from this model versus the quantiles of a t_3 distribution indicates that the t_3 distribution has too heavy tales. Also a t_4 distribution has too heavy tales, whereas a t_5 distribution seems to be suitable. We therefore fix ν to 5, and estimates the other parameters by maximum likelihood (shown in the “estimate t” column of Table 5). The left panel of Figure 7 shows the quantiles of the standardized residuals versus the quantiles of a t distribution with 5 degrees of freedom. These plots confirms that the t distribution fits well. We therefore conclude that the model (7) modified by (8) and (9) is a reasonable good model.

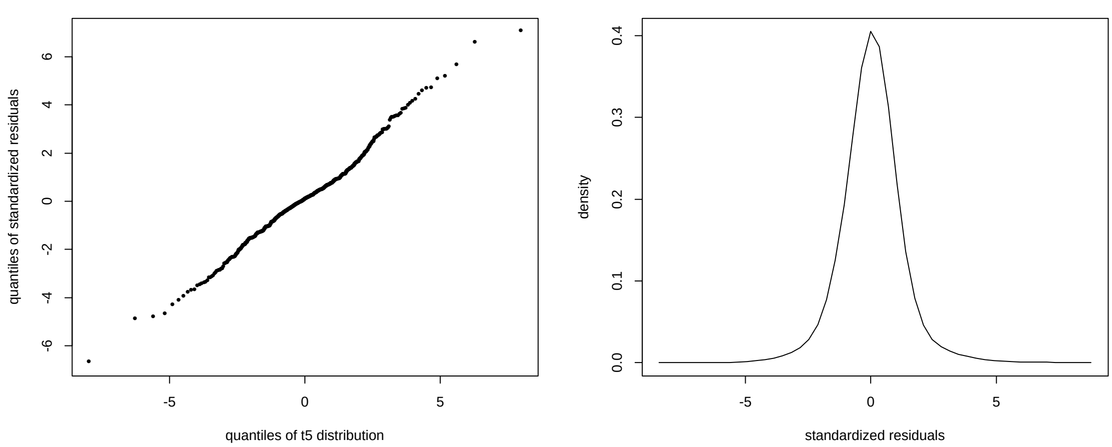

Figure 7 Left panel: Quantiles of standardized residuals from model (7)- (9) versus quantiles from a t_5 distribution. Right panel: A smoothed density plot of the standardized residuals.

To summarise, we propose the following measurement error model for the angles, based on the buoy data:

$$\Theta = 1.045 \cdot \theta + \varepsilon(\theta) \quad ,$$
$$\varepsilon(\theta) = s(\theta) \cdot t_5 \quad \text{, where } t_5 \text{ is t distributed wth 5 degrees of freedom,} \tag{10}$$
$$s(\theta) = \exp(1.057 + 0.00918 \cdot \min(\text{abs}(\theta), 55))$$
$$= 2.88 \cdot \exp(0.00918 \cdot \min(\text{abs}(\theta), 55)) \quad .$$

If one want a model with Gaussian errors, one may alternatively use

$$\Theta = 1.057 \cdot \theta + \varepsilon \quad ,$$
$$\varepsilon(\theta) \sim N(0, \sigma^2(\theta)) \quad , \tag{11}$$
$$\sigma(\theta) = \exp(1.336 + 0.0117 \cdot \min(\text{abs}(\theta), 55))$$
$$= 3.80 \cdot \exp(0.0117 \cdot \min(\text{abs}(\theta), 55)) \quad .$$

The standard deviations of the models (10) and (11) are shown in Table 6.

Table 6 Standard deviations models (10) and (11).

θ	0	15	30	45	55
$\sigma(\theta)$ of model (10)	3.7	4.2	4.9	5.6	6.2
$\sigma(\theta)$ of model (11)	3.8	4.5	5.4	6.4	7.2

5 Dependency among observations from the two platforms

So far, we have made models for the observed radial distances and angles from one platform, without considering possibly dependencies among the pairwise observations from the two platforms on the same buoy.

Let R_A and R_B be observed radial distances from platform A and B, respectively, on the same whale, and let $\varepsilon_A(r)$ and $\varepsilon_B(r)$ denote the corresponding errors from model (4). Assume that the errors can be decomposed into one component separate for each platform and one component common for both platforms, i.e.

$$\varepsilon_A(r) = \nu_A(r) + \omega(r) \qquad , \qquad \varepsilon_B(r) = \nu_B(r) + \omega(r) \quad , \tag{12}$$

with $Var(\nu_A(r)) = Var(\nu_B(r)) = \gamma^2(r)$ and $Var(\omega(r)) = \varphi^2(r)$, and thus $\sigma^2(r) = \gamma^2(r) + \varphi^2(r)$.

Ignoring the dependency of r for a while, $\text{Var}(\log(R_A) - \log(R_B)) = 2\gamma^2$ is a measure of the random variation between two platforms observing the same whale. We have estimated γ from the 820 pairs of (R_A, R_B) in the present buoy data, which gave $\hat{\gamma}_b = 0.311$. (The subscript *b* denotes estimates based on buoy data, to distinguish them from estimates based on real sightings data, which will be discussed in the next section). Note that this estimate is an overall estimate for the whole data set, averaged over all true distances. This can be compared to the estimate of σ from model (1), which estimates the standard deviation of $\varepsilon(r)$, averaged over all distances *r* in the buoy data. The estimate of σ is $\hat{\sigma}_b = 0.388$ (see Table 1). Thus, the random variation between two observers observing the same whale constitutes only 64% ($0.311^2/0.388^2$) of the total variance. The remaining variance (φ^2) may for instance be due to variation in the weather conditions. If we assume that $\gamma^2(r)$ and $\varphi^2(r)$ are proportional to $\sigma^2(r)$ for all *r*, i.e. $\gamma^2(r) = c^2 \cdot \sigma^2(r)$ and $\varphi^2(r) = (1-c^2) \cdot \sigma^2(r)$, we then get the estimate $\hat{c} = \hat{\gamma}_b / \hat{\sigma}_b = 0.80$, which yields $\gamma^2(r) = 0.64 \cdot \sigma^2(r)$ and $\varphi^2(r) = 0.36 \cdot \sigma^2(r)$.

We then do a corresponding analysis for the variability in the angles. Let Θ_A and Θ_B be the two observed angles on the same whale, and assume that the error $\varepsilon(\theta)$ in (7) or (8) can be decomposed as in (12), but with *r* replaced by θ. We have computed the estimate of $\mathrm{Var}(\Theta_A - \Theta_B) = 2\gamma^2$ from the 820 pairs of (Θ_A, Θ_B) in the buoy data (ignoring the dependency of θ). This gave $\hat{\gamma}_b = 4.05$, which is much lower than the corresponding estimate of σ from model (6), which is 5.41 (Table 4). Thus only about 56% of the variance of the measurement error is related to random variation between the two platforms. the corresponding estimate of *c* is $\hat{c} = \hat{\gamma}_b / \hat{\sigma}_b = 0.75$, which yields $\gamma^2(r) = 0.56 \cdot \sigma^2(r)$ and $\varphi^2(r) = 0.44 \cdot \sigma^2(r)$.

6 Re-estimating variability from sightings data

The buoy experiments have been designed as close to the real sighting situations as possible, but there will of cause be some difference. One may therefore argue that one should use sightings data to estimate the bias and variability in estimates of the radial distance, if possible. Concerning the bias, this is obviously not possible. However, Schweder (1997) suggested estimating the variability from the sightings data.

We first consider the radial distances. We have estimated γ from 109 duplicates from the 1996-2001 sightings data, which gave $\hat{\gamma}_s = 0.334$. This value can be compared to the corresponding value from the buoy data given in Section 5, $\hat{\gamma}_b = 0.311$. (The subscripts *s* and *b* denote buoy and sightings data, respectively). The slightly higher value of $\hat{\gamma}_s$ compared to $\hat{\gamma}_b$ can be explained by the mean radial distance being slightly less in the sightings data, i.e. we can expect higher mean variance in the sightings data. We conclude that the estimated values of γ from the two data sets are in accordance with each other, and we therefore suggest using the buoy data to estimate both the bias and the variability in observed radial distances.

Next, we consider the angles. We have estimated γ from the 109 duplicates in real sightings in the 1996-2001 surveys, which gave $\hat{\gamma}_s = 8.13$. This is double the corresponding estimate from the buoy data given in Section 5, $\hat{\gamma}_b = 4.05$, which strongly indicates that the observation process is different in the buoy experiments than in the real sightings. However, a plot (not shown) of the quantiles of $\Theta_A - \Theta_B$ from the buoy data versus those from the survey data shows a linear relationship, indicating that the distributions have the same shape, but different variance.

We therefore suggest that the structural forms of the variances of $\varepsilon(\theta)$ in (7) and (8) are kept, but that the levels are modified according to the real sightings data. First, we must take into account that the mean observed (absolute) angle in the buoy data is 30 degrees, whereas it is 45 degrees in the real sightings data. Using the t-distribution model, Table 6 shows that $\sigma(\theta)$ increases from 5.4 to 6.4 when θ increases from 30 to 45 degrees. If we, as in Section 5, assume that $\gamma^2(\theta)$ and $\varphi^2(\theta)$ are proportional to $\sigma^2(\theta)$, then $\gamma(\theta)$ will also increase by a factor 6.4/5.4 when θ increases from 30 to 45 degrees. Therefore, the factor $\hat{\gamma}_s/((6.4/5.4) \cdot \hat{\gamma}_b) = 1.69$ is a fair estimate of how much higher $\gamma(\theta)$ is in the real sightings compared to the buoy experiments. We assume that this factor holds for all values θ and for $\varphi(\theta)$ as well, and thus also for $\sigma(\theta)$.

7 Proposed models relevant for real sightings

Finally, we present our proposed models, that we find relevant to model observational errors in the real sightings. According to the discussion in Section 5, the errors are divided into one component different for each platform, and another component common to both platforms. Furthermore, the variance of the angle model is adjusted, so it is in accordance with the estimated value of γ^2 from the real sightings data.

The model for the radial distances observed from platform A is

$$R_A = \exp(-0.108) \cdot r \cdot \exp(\varepsilon_A(r)) = 0.898 \cdot r \cdot \exp(\varepsilon_A(r)) \quad ,$$
$$\varepsilon_A(r) = \nu_A(r) + \omega(r) \quad ,$$
$$\nu_A(r) \sim \mathrm{N}(0, 0.64 \cdot \sigma^2(r)) \quad , \tag{13}$$
$$\omega(r) \sim \mathrm{N}(0, 0.36 \cdot \sigma^2(r)) \quad \text{, independent of } \nu_A(r) \quad ,$$
$$\sigma(r) = \exp(0.542 - 0.237 \log(r)) = 1.71 \cdot r^{-0.237} \quad .$$

The model for platform B is similar, with a separate error component $\varepsilon_B(r)$, but a common error component $\omega(r)$.

The model for the angles observed from platform A, with Gaussian errors, is

$$\Theta_A = 1.057 \cdot \theta + \varepsilon_A(\theta) \quad ,$$
$$\varepsilon_A(r) = \nu_A(r) + \omega(r) \quad ,$$
$$\nu_A(r) \sim \mathrm{N}(0, 0.56 \cdot \sigma^2(r)) \quad , \tag{14}$$
$$\omega(r) \sim \mathrm{N}(0, 0.44 \cdot \sigma^2(r)) \quad \text{, independent of } \nu_A(r) \quad ,$$
$$\sigma(\theta) = 1.69 \cdot \exp(1.336 + 0.0117 \cdot \min(\mathrm{abs}(\theta), 55))$$
$$= 6.42 \cdot \exp(0.0117 \cdot \min(\mathrm{abs}(\theta), 55)) \quad .$$

Again, the model for platform B is similar, with a separate error component $\varepsilon_B(r)$, but a common error component $\omega(r)$.

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