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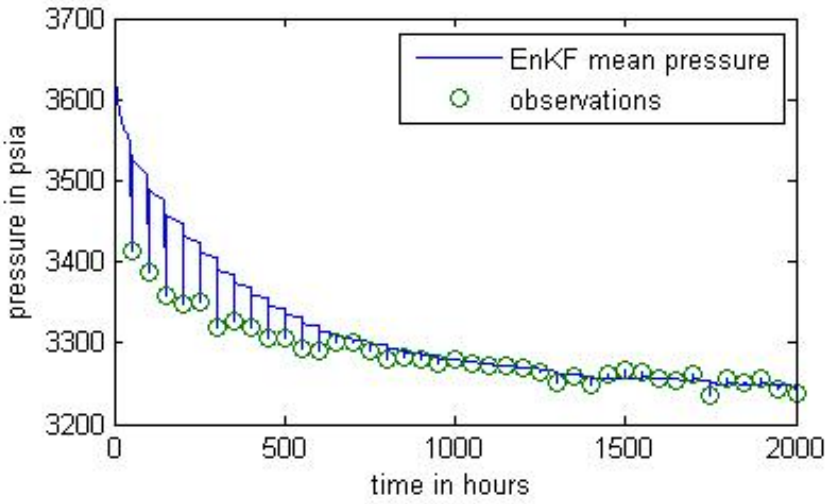
Note

The Ensemble Kalman Filter - theory and applications in oil industry



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Abstract

We give an introduction to the Kalman filter and the Ensemble Kalman filter. The method is illustrated with the help of synthetic examples. We also give a review of the applications of Ensemble Kalman filter into the field of history matching of oil reservoirs.

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Contents

1 Introduction

The purpose of this note is to give a practical introduction into Kalman filtering and one of its byproducts, the Ensemble Kalman Filter. The motivation for this study is that the Ensemble Kalman filter has become a popular method for doing history matching of oil reservoirs.

In 1960, Rudolph Kalman published the paper Kalman (1960) where he made a major contribution to Stochastic Control Theory, namely, Kalman filter. Engineers working in navigation quickly realized that the Kalman filter gave a solution to a number of problems that were previously intractable with the use of the Wiener filter. The Kalman filter was timely, it was discovered at a time when significant progress was achieved in the area of digital computers. The circumstances were perfect for testing the computationally demanding operations required by the recursive nature of Kalman's algorithm.

The filter was originally designed to give a solution to the classical linear least squares estimation problem in signal processing and control theory. Unfortunately, it took over 20 years to incorporate the Kalman filter into mainstream statistics, despite the straightforward interpretation as a recursive Bayesian estimator. Today the Kalman filter is covered by books in time series like West and Harrison (1997).

The filter may be summarized as follows: It is a set of equations to estimate the state of some process, with the possibility of assimilating new information as it arrives. The filter is optimal in a sense that it incorporates all recently acquired information in the best possible way. The information arrives typically when we measure the state variables of the process.

The main ideas are explained with the help of examples. We also review recent work done in this field so as to give a flavor of the state of the art in this technique.

1.1 Lost at sea

This instructive example was extracted from Chapter 1 in Maybeck (1979).

You are lost at sea, at night, and you have no idea of your location. Call your location x (small letters denote both stochastic variables and realizations) and assume that we are in a one-dimensional sea. You want to stablish your position with the help of a star sighting. Unfortunately, you do not have much experience with measuring the position, so your measurement is very inaccurate: you get the position z_1 but with some large uncertainty σ_1. First detail: it is good to model the process with some sort of uncertainty. Namely, in the sea, the waves make the boat balance back and forth bringing uncertainty to the 'real' position. In probabilistic terms this is known as a random walk.

Fortunately, you are sailing with a good friend that has more experience with finding the location. Immediately after you made your measurement, he made his decision and declared that the position should be z_2. He has less uncertainty in his result, therefore we assign him a standard deviation σ_2, which is less than σ_1.

The problem now is naturally: estimate the position of your boat given these two measurements. We want to use all the information available, and combine them in the best possible way.

We can put this problem in mathematical terms. Consider the variable $x^* = x|z_1$ and assume that $x^* \sim N(z_1, \sigma_1)$. This is the so-called prior and provides some a-priori information on the position. The likelihood function of the second observation is given by $z_2|x^* \sim N(x^*, \sigma_2)$, namely, the effect of adding new information is given some weight.

We are interested in the moments μ and σ of $f_{x|z_1,z_2}$. At this point we should note one of the essential points in Kalman estimation: the estimate μ should be a linear blend of the prior x^* (all past information) and the newly arrived information.

Bayes theorem states that the posterior $f_{x^*|z_2}$ is proportional to $f_{z_2|x^*} f_{x^*}$. This amounts to

saying that

$$(x^* - \mu)^2/2\sigma^2 = (z_2 - x^*)^2/2\sigma_2^2 + (x^* - z_1)^2/2\sigma_1^2 + C \tag{1}$$

for some constant C. The following formulas now follow by comparing the coefficients in $(x^*)^2$ and x^*, respectively:

$$\mu = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)z_1 + \sigma_1^2/(\sigma_1^2 + \sigma_2^2)z_2 \tag{2}$$

$$1/\sigma^2 = 1/\sigma_1^1 + 1/\sigma_2^2 \tag{3}$$

We expect something like the illustration of Figure 1.

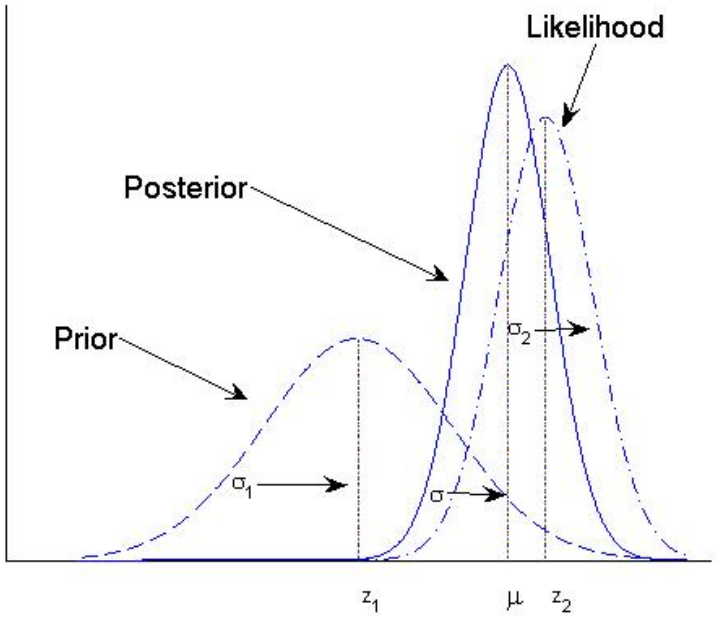

Figure 1. Assimilation of information and Bayes rule.

The estimated position μ is, as expected, a weighting average between the measurement taken by you and your friend. Since your friend is better navigator, a larger weight is given to his/her measurement z_2. On the other hand, the variance σ^2 is proportional to the harmonic mean of the variances in the observations. In particular, the new standard deviation is less than both! This is an important remark: the variance of the "update step" decreases.

A good notation could be:

- $x^f = E[x|z_1] = z_1$: the average state of the process, given the existing information/measurement (your poor measurement)
- $x^a = E[x|z_1, z_2] = \mu$: updated average state of the process, given the newly arrived information (thanks to your friend)
- $(\sigma^f)^2 = \mathrm{var}(x|z_1) = \sigma_1^2$, your rather large variance
- $(\sigma^a)^2 = \mathrm{var}(x|z_1, z_2)$, the updated variance

With this notation, the update step (see (2)) is a linear combination of the average state and the new measurement:

$$x^a = x^f + K[z_2 - x^f] \tag{4}$$

where $K := (\sigma^f)^2/((\sigma^f)^2+(\sigma^a)^2)$ is commonly known as Kalman gain. This is the first important equation of the Kalman update (or analysis).

Likewise, the uncertainty (see (3)) may be written as:

$$(\sigma^a)^2 = (\sigma^f)^2 - K(\sigma^f)^2 = (1-K)(\sigma^f)^2. \tag{5}$$

And this is the second equation of the Kalman update. We are now done with the static problem.

Your friend is now alone in the boat and he knows that he is moving according to the following model:

$$dx/dt = u + w. \tag{6}$$

In other words, u is the velocity of the vessel, and w is some noise term (white Gaussian) that models the uncertainty in the knowledge of the actual velocity due for example to disturbances, changes in wind speed, etc. Our intention is to replace the role of your bad observation by a "forecast" of the average of your friend's observation at the initial time. Now you are the mathematician again and this is your goal: "project forward in time" the available estimate until your friend takes another look at the stars tomorrow. That is, you let the model forecast the position tomorrow. The average position has moved according to the motion law, and the uncertainty in the position tomorrow will become larger than today.

A discrete interpretation of the law of motion with Euler time stepping is simply

$$x^t = x_0 + \Delta t u + \Delta t w_0 \tag{7}$$

where the position today $x_0 \sim N(z_0, \sigma_0)$ and $w_0 \sim N(0, \sigma_w)$. The superscript "t" means "true state of the process according to the model".

Equation (7) moves the state of the process, along with the moments. One derives the equations:

$$x^f = z_0 + \Delta t u \tag{8}$$

$$(\sigma^f)^2 = \sigma_0^2 + \Delta t \sigma_w^2 \tag{9}$$

Note the increase in variance of σ^f! These are now playing the role of your inaccurate observations and becomes the a-priori information.

Your friend is now trying to draw some information from the true state:

$$z = x^t + v \tag{10}$$

where $v \sim N(0, \sigma)$. He/She is just measuring x^t, and no other function of the position. In general Kalman filtering one may allow for a so-called observation operator to affect the state process.

We may now write the update equations:

$$x^a = x^f + K_1[z_1 - x^f] \tag{11}$$

$$(\sigma^a)^2 = (1-K_1)(\sigma^f)^2 \tag{12}$$

where the Kalman gain is $K_1 = (\sigma^f)^2/((\sigma^f)^2+\sigma^2)$. The process is restarted with the moments x^a and σ^a of the initial state x_0.

1.2 The oil connection

One may now recognize the philosophy of Kalman filtering. The following two steps are distinguished:

1. Project forward in time the current state to obtain the a-priori estimate.

2. Incorporate the new measurement and obtain an improved a-posteriori estimate.

How do one applies these ideas into oil reservoir monitoring? Say that you are interested in production variables evolving in time like gas/oil ratio in some wells. Say that you know the permeability and porosity today. The first step above is carried out by some flow simulator. Obtain some forecast of tomorrow's state of the reservoir given today's data. Take a new measurement tomorrow and update the a-priori estimate. Proceeding this way, one intends to mimic the behavior of the reservoir. Note that in general, the simulators are highly non-linear, so Kalman filter will break down. The use of the Ensemble Kalman Filter (EnKF) has proved useful in tackling with these problems, see Nævdal et al. (2005). We will come back to this later.

2 The discrete Kalman Filter

A very friendly exposition of the Kalman filter is found in the book Brown and Hwang (1992). For a summary of the problem and the equations we also recommend Welch and Bishop (2004).

2.1 The problem

Throughout the literature one encounters that $x_k \in \mathbb{R}^n$ denotes the stochastic process itself. This is to distinguish the process and observations from matrices that are denoted by capital letters. So, assume that some process is governed by the linear stochastic difference equation:

$$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1}, \quad k \geq 1. \tag{13}$$

with initial state

$$x_0 = \xi \sim N(x_0, P_0) \tag{14}$$

This process is measured and one gets m possible measurements, namely, $z_k \in \mathbb{R}^m$:

$$z_k = Hx_k + v_k. \tag{15}$$

The initial state, the process noise w_k and the measurement noise v_k are mutually independent. The noises have the law

$$w_k \sim N(0, Q_k) \tag{16}$$

$$v_k \sim N(0, R_k) \tag{17}$$

(18)

Matrices A and B relate the actual state of the process with a previous state and control process, respectively. Matrix H is a so-called measurement operator. Note that both the process and the measurements are described by linear models. In practice models like this are too restrictive. One of the main applications of the Kalman filter is actually into non-linear models.

2.2 Kalman filter equations

Consider the problem:

$$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1} \tag{19}$$

$$z_k = Hx_k + v_k \tag{20}$$

Here x_{k-1} denotes the state at time t_{k-1} and x_k denotes the state at some later time t_k. Let x_k^f denote the a-priori estimate of the process at this later point based on all our knowledge prior to

this point, in mathematical terms:

$$x_k^f = E(x_k|z^*) \tag{21}$$

where z^* denotes the available information up to time t_{k-1}. This is our best guess prior to assimilating the measurement z_k.

Let x_{k-1}^a means the best estimate of the process with assimilated information up to t_{k-1}, i.e.,

$$x_{k-1}^a = E(x_{k-1}|z^*) \tag{22}$$

Equation (19) is used to obtain the forecast estimate at time t_k. Conditioning on the observations z^* gives:

$$x_k^f = Ax_{k-1}^a + Bu_{k-1} \tag{23}$$

The forecast covariance matrices are updated according to the matrix relation:

$$P_k^f = AP_{k-1}^a A^t + Q_{k-1}. \tag{24}$$

where

$$P_k^f = E\left[(x_k^f - x_k)(x_k^f - x_k)'\middle| z^*\right], \tag{25}$$

$$P_{k-1}^a = E\left[(x_{k-1}^a - x_{k-1})(x_{k-1}^a - x_{k-1})'\middle| z^*\right]. \tag{26}$$

Once we have obtained the moments of the prior distribution, we are able to assimilate new information. The assimilation/analysis step is carried out as follows. The a-posteriori mean is given by:

$$x_k^a = x_k^f + K_k(z_k - Hx_k^f) \tag{27}$$

where

$$K_k = P_k^f H'(HP_k^f H' + R_k)^{-1} \tag{28}$$

is the Kalman gain matrix. The a-posteriori error covariance matrix is computed by the formula

$$P_k^a = (I - K_k H)P_k^f. \tag{29}$$

Equations (23), (24), (27), (28) and (29) constitute the Kalman filter equations.

2.3 Examples

We will now illustrate the behavior of the filter on two one dimensional examples.

2.3.1 Estimation of a random constant

This is a simple, yet important example. The idea is to show that, for the problem of estimating a random constant with infinity a-priori variance, the Kalman estimation gives the usual estimate $\bar{z}$, i.e., the mean of the observations.

The model is simply:

$$\begin{aligned} x_k &= x_{k-1}, \\ z_k &= x_k + v_k. \end{aligned}$$

The variable to estimate is $x_0 \sim N(\mu_0, \sigma_0^2)$ and the error noise is $v_k \sim N(0,1)$. So, assuming $Q_k = 0$ is to keep this variable static, and $R_k = 1$ is the observation variance.

We only carry out the two first iterations.

k=1:

$$\mathbf{F}: \left\{ \begin{array}{lcl} x_1^f &=& x_0^a \\ P_1^f &=& P_0^a = \sigma_0^2. \end{array} \right.$$

Assuming $\sigma_0 = \infty$ we proceed to assimilate the first observation:

$$\mathbf{A}: \left\{ \begin{array}{lcccl} K_1 &=& \frac{P_1^f}{P_1^f+R_1} &=& 1 \\ x_1^a &=& x_1^f + K_1(z_1 - x_1^f) &=& z_1 \\ P_1^a &=& (1-K_1)P_1^f &=& \frac{R_1 P_1^f}{P_1^f+R_1} = 1 \end{array} \right. .$$

The equalities to 1 are in sense of taking the limit when $P_1^f \to \infty$. We may go on to compute the second step.

k=2:

$$\mathbf{F}: \left\{ \begin{array}{lcl} x_2^f &=& x_1^a = z_1 \\ P_2^f &=& P_1^a = 1. \end{array} \right.$$

$$\mathbf{A}: \left\{ \begin{array}{lcccl} K_2 &=& \frac{P_2^f}{P_2^f+R_2} &=& 1/2 \\ x_2^a &=& x_2^f + K_2(z_2 - x_2^f) &=& z_1 + \frac{1}{2}(z_2 - z_1) = \frac{z_1+z_2}{2} \\ P_2^a &=& (1-K_2)P_2^f &=& 1/2 \end{array} \right.$$

2.3.2 Random walk

This example is essentially the "lost at sea" example from the introduction, namely,

$$\begin{aligned} x_k &= x_{k-1} + w_{k-1}, \\ z_k &= x_k + v_k. \end{aligned}$$

We take $Q_k = 1$ for $k \leq 10$ and $Q_k = 1/2$ for the rest. The observation errors have variance $R_k = 1/2$. In figure 2 you may see how the arrival of observations influence the filter and how this is able to reproduce the true $x-$process shown as a solid line.

3 Ensemble Kalman Filter

We drop the indices, and concentrate on the iterative nature of the filter. The Kalman filter is designed for linear models. Consider a nonlinear model like:

$$x^t = f(x) + w \tag{30}$$

$$z = Hx^t + v \tag{31}$$

for some non-linear function f of the state. An early attempt to extend the Kalman filter to these problems is the Extended Kalman filter, which is based on linearization of the nonlinear model with the tangent operator (Jacobian $F := \frac{\partial f}{\partial x}$). However, the extended Kalman filter is not suitable for very large models, and fails if the nonlinearities are too severe.

The Ensemble Kalman Filter (EnKF) was introduced by Evensen (1994) to handle nonlinear ocean models, and has shown to be a promising approach.

Given the nonlinear character of the model dynamics, the moments of the real state x^t are in general difficult to find, see (25). We resort to simulation. The idea is to start with a set or ensemble

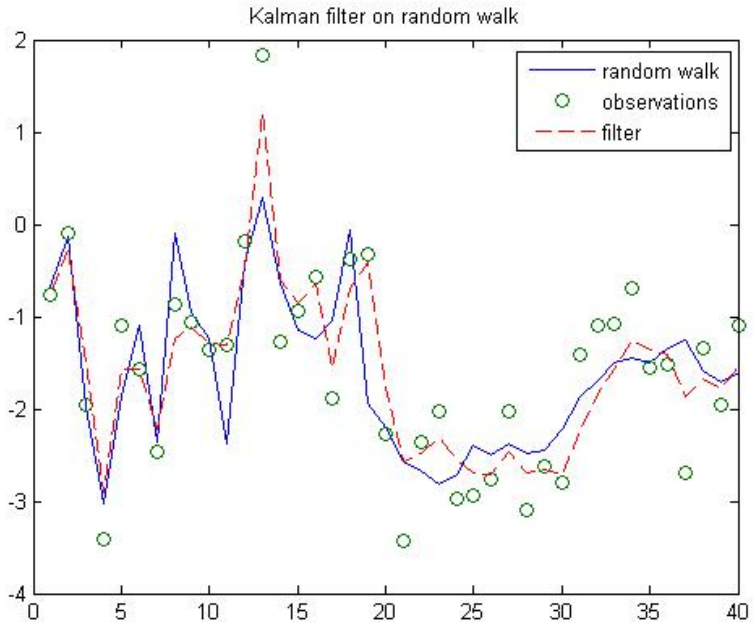

Figure 2. Random walk example.

of model states (we postpone the simulation details), evaluate each realization of the model state through the model dynamics (30) and obtain a new ensemble. This is the a-priori ensemble, from which statistics can be extracted from usual estimation. The update step is done now, again with all the members in the ensemble, but we simulate this time an ensemble of observations, taking as mean the true observations. Each of the a-priori ensemble member is then updated to reflect the simulated observations. The process is then repeated to hope for convergence.

In few words, the EnKF is a Monte Carlo type Kalman Filter.

The true state of the process is approximated by the mean of the members of the ensemble. This substitution has no effect in the form of the equations.

3.1 Notation

Here is a short overview of the notation we use for the ensemble Kalman filter. Note that the time index is still omitted. The notation and equations are the same as in Evensen (2003).

- $X := [x_1, \ldots, x_N] \in \mathbb{R}^{n \times m}$ matrix of ensemble members. Each state vector x_i is in $\mathbb{R}^n$.
- $\bar{X} := X[1, \ldots, 1]'_N \in \mathbb{R}^{n \times 1}$ the ensemble mean vector. The elements of the ensemble mean are the mean of the rows of matrix X.
- $z \in \mathbb{R}^{m \times 1}$: vector of observations
- $P_e := \frac{(X - \bar{X})(X - \bar{X})'}{N-1} \in \mathbb{R}^{n \times n}$: the ensemble covariance matrix.
- $Z := z + \Upsilon \in \mathbb{R}^{m \times N}$ matrix of perturbed observations
- $\Upsilon := (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{R}^{m \times m}$ vector of perturbations where $\epsilon_i \sim N(0, \Sigma_i^e)$, and Σ_i^e is a diagonal matrix.
- $R_e = \frac{\Upsilon \Upsilon'}{N-1}$

3.2 EnKF Algorithm

The main idea to bear in mind is that the EnKF is a Monte Carlo Kalman filter. Simultaneous state vectors are simulated, advanced in time, and updated for each time an observation arrives. The statistics of the forecast step are not available explicitely, as in the linear case. Hence the need for simulation. The statistics are estimated from the ensemble, namely $\bar{X}$ and P_e. As the number of ensemble members increases, these estimates will converge towards the true first and second moments of a-priori distribution after the forecast step. The EnKF algorithm is given below.

1. **Initialize:** Define an initial ensemble $X \sim N(\mu_0, \Sigma_0)$

2. **Forecast:** Project forward in time the ensemble X using the equation:

$$X^f = f(X) + W \tag{32}$$

   where the model noise matrix $W = (w_i)$ is make by drawing the random numbers $w_i \sim N(0, \Sigma_i^w)$. Repeat this step until you reach an observation time.

3. **Analysis:** Generate observations Z by perturbing the real observations z. Update the ensemble X^f using the equation

$$X^a = X^f + P_e H'(HP_eH' + R_e)^{-1}(Z - HX^f). \tag{33}$$

   Go to the **Forecast** step.

The algorithm stops at some user defined time T. One of the tricky parts of the algorithm is precisely the inversion of the matrix $HP_eH' + R_e$ in the analysis step. A detailed discussion of this issue is found in Evensen (2003). We only mention that for oil applications, where the number of observations m is large, the inverse maybe computed at a cost proportional to mN (recall that N is number of samples in the ensemble) instead of m^2.

It is worth mentioning that the EnKF belongs to the family of so-called particle filters or Monte Carlo filters. These filters appeared in the 90's and turned out to produce better results than the Extended Kalman Filter. For a nice set of examples and references see Kitagawa (1998), see also the nice tutorial Arulampalam et al. (2002). The EnKF is designed for large-scale filtering problems.

3.3 Examples

We have implemented two examples in order to illustrate the algorithm. We start by considering a synthetic history matching problem, where the fluid simulator is a simple radial flow simulator, for which analytic solutions are known. The parameters and equations are from Ahmed and McKinney (2005). The second example is taken from Evensen (2003).

3.3.1 Well test equation

We consider an example of radial flow into a well bore. In absence of reservoir heterogeneities, the flow into or from a well will follow radial flow lines a substantial distance from the well bore. Figure 3 shows idealized flow lines for a radial flow system. In this example, oil is flowing through a porous medium. The equation governing the flow is normally written in terms of the pressure. Since we assume radial symmetry, the pressure P depends only on the radius r and time t. The equation reads:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) = \frac{\phi\mu c}{0.0002637\kappa}\frac{\partial p}{\partial t}. \tag{34}$$

This is the so-called diffusivity equation and it is considered one of the most important mathematical expressions in petroleum engineering. This equation is derived under the assumption that the

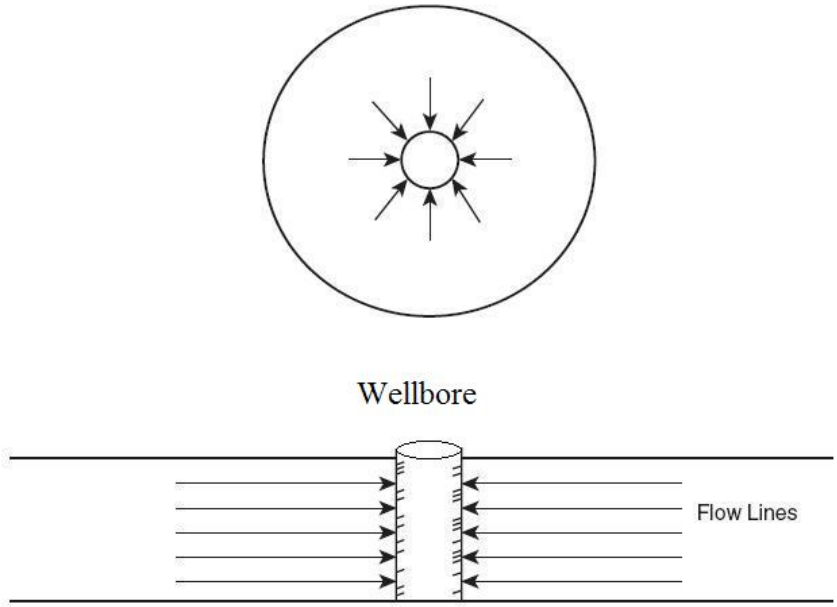

Figure 3. Plan and side view of flow into a well bore.

permeability and viscosity are constant over pressure, time and distance ranges. The fluid is assumed to be slightly compressible, like, say, oil. The notation $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial t}$ means partial derivative with respect to r and t respectively. The parameters with their units of measurement are:

p : pressure in psi
r : radius in ft
t : time in hours
κ : permeability in md
μ : viscosity in cp
ϕ : porosity in percent
c : total compressibility in psi^{-1}

The constant 0.0002637 is just a conversion factor from, say, feet into cm, darcies into milidarcies, etc. The diffusivity equation is defined in a reservoir that is a cylinder with a hole. A slice of the reservoir, illustrating the different parameters from the geometry is provided in Figure 4. These are the relevant parameters:

r_e : external radius
r_w : well radius
pb : pressure at well surface (bottom-hole pressure)
h : thickness of the reservoir

A parabolic differential equation requires an initial condition and boundary conditions. The initial condition is the initial state of the pressure before the well is opened. So, we let $p(r, 0) = p_i$. Immediately after opening the well, the well pressure drops and fluid starts flowing into the well. This is caused by a difference in pressure, i.e., the fluid moves from high pressure regions into

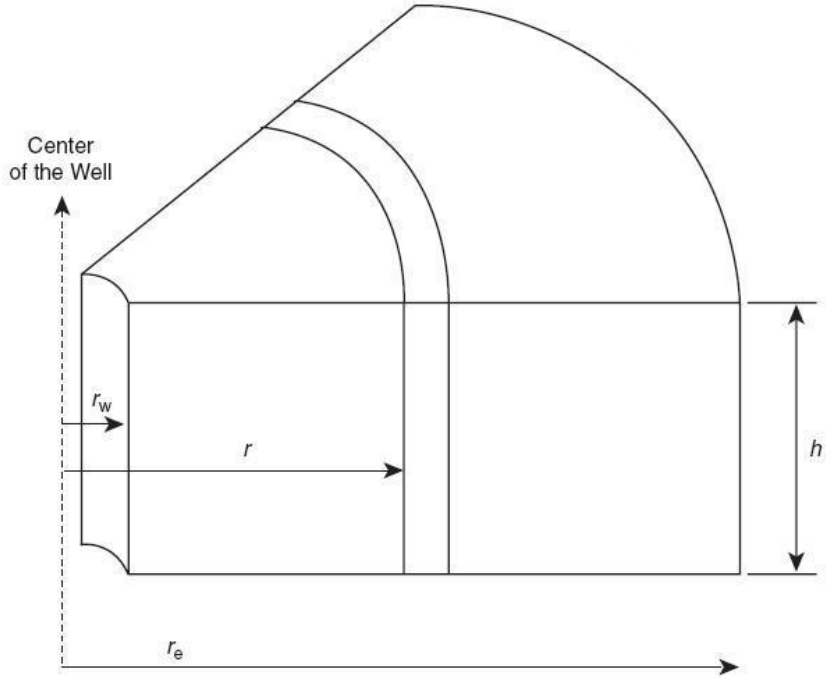

Figure 4. Reservoir geometry.

low pressure regions. One of the ideal assumptions in order to obtain a closed form solution is to impose a Dirichlet boundary condition at $r_e = \infty$, namely, we assume and infinite reservoir and let $p(\infty, t) = p_i$. It now remains to impose a boundary condition on the wall of the well. Essentially, there exists two kinds of boundary conditions: Dirichlet and Neumann. In this case we are interested in controlling the flow rate q into the well. This is typically a Neumann boundary condition. Darcy's law comes to rescue us in the form:

$$q = 0.001127 \frac{2\pi r h \kappa}{\mu} \frac{\partial p}{\partial r}|_{r=r_w}. \tag{35}$$

Is it customary to express the flow rate as $q = B_0 Q_0$, where B_0 represents the oil formation volume factor in bbl/STB (well field barrels/ stock tank barrels) and Q_0 is the oil flow in STB/day. The solution of the diffusivity equation is given by the expression:

$$p(r,t) = p_i - \left[\frac{70.61 Q_0 \mu B_0}{\kappa h}\right] \mathrm{Ei}\left[\frac{948 \phi \mu c r^2}{\kappa t}\right] \tag{36}$$

where Ei is the so-called exponential integral defined as:

$$\mathrm{Ei}(x) = \int_x^\infty \frac{e^{-u}}{u} du \qquad \text{for } x > 0. \tag{37}$$

EnKF

We are interested in estimating the permeability based on observations of the bottom-hole pressure. So, our state vector consists of two variables, the static variable κ and the dynamic variable pb. The theoretical bottom hole pressure is obtained from evaluating (36) at the well radius $r = r_w$. The observation vector has one component, namely, the observed pb. In our notation:

$$\begin{aligned} x &= [\kappa,\ pb]' \qquad \text{state vector} \\ z &= pb_{obs} \qquad \text{observations} \end{aligned}$$

We would like to model the permeability as static, thus, we assume that κ_0 is drawn from a normal distribution with some a-priori mean and variance. Subsequent κ_k are modeled as:

$$\kappa_k = \kappa_{k-1} \tag{38}$$

where κ_k represents the permeability at time $t_k = k\Delta t$. Note that no model error is assumed for the permeability.

On the other hand, our simulator of pressures provides us with the following model for the bottom-hole pressure:

$$pb_k = p(r_w, t_k, \kappa_k) + w_{k-1} \tag{39}$$

Under general circumstances, the simulator is a PDE finite difference solver that takes into account the previous state of pressures for all spatial coordinates to advance in time the pressure. In this example, a closed formula is available that gives us a dependence only on the permeability, and not on previous pressures.

The observations can be written as follows: $z_k = Hx_k + v_k$, where the operator H is the projection of the state vector, namely, $H = [0\ ,\ 1]$.

We start by defining a time interval from time $t = 0$ until time T. Inside this interval we have simulated a set of equally spaced observations by employing equation (39), namely, given κ_{true}, we perturb the pressure at observation times t_{obs} given by $p(r_w, t_{obs}, \kappa_{true})$ with some Gaussian noise. The objective is to show that the filter will find the true permeability after a number of assimilations.

Experiment

The numerical experiments in this section are intended to show how the EnKF algorithm may be used to recover the true permeability. The experiment is set up for $T = 2000$ hours, with the following parameters

$$\begin{aligned} r_w &= 0.1 \text{ ft} \\ \mu &= 1.5 \text{ cp} \\ Q_0 &= 300 \text{ STB/day} \\ B_0 &= 1.25 \text{ bbl/STB} \\ \phi &= 15\ \% \\ p_i &= 4000 \text{ psi} \\ h &= 15 \text{ ft} \\ c &= 12e-6 \text{ psi}^{-1} \end{aligned}$$

The time steps are half of an our, and the number of observations is 40, equally spaced and represented by circles in Figure 5. The initial ensemble is drawn with a mean permeability of 80, and the true permeability is 60. Observe how the observations will eventually correct the permeability in 7, and the reduction in standard deviation in 6. From Figure 8 it is clear that at each assimilation there is a drop in the standard deviation of the ensemble, as expected.

Finally, in Figure 9 we plot the probability density functions just before we carry out the assimilation of the pressure measurement number 5 (forecasted) and just after. The updated pdf gets closer in mean to the observation at this time.

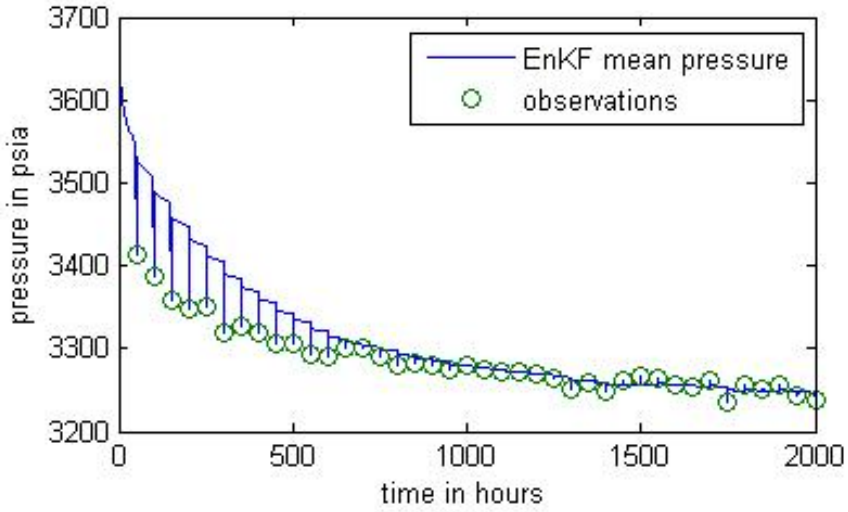

Figure 5. EnKF mean and observations for pressure.

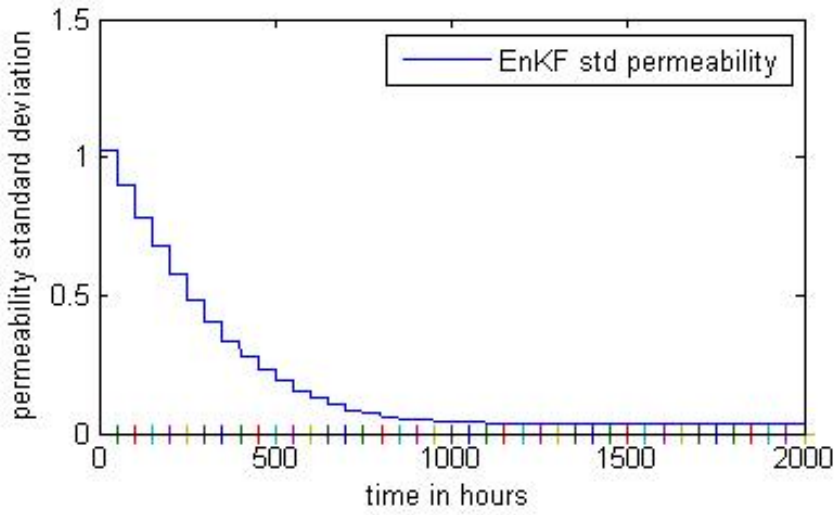

Figure 6. EnKF standard deviation for permeability.

3.3.2 Scalar example

This example is extracted from Evensen (2003). This is a linear example, where the member ψ of the state vector is model by the a red-noise. The state vector follows the law:

$$\begin{bmatrix} q_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} \alpha q_{k-1} \\ f(\psi_{k-1}) + \sqrt{\Delta t}\sigma\rho q_k \end{bmatrix} + \begin{bmatrix} \sqrt{1-\alpha^2} w_{k-1} \\ 0 \end{bmatrix} \tag{40}$$

The function f is chosen to be $f(\psi) = \psi$ for the linear case. The parameter ρ is chosen to ensure that the variance growth becomes independent of α and Δt in the linear case. Observe that the form of (40) is the one we are used to. The red-noise behaviour is controlled by the parameter α. For $\alpha = 0$ the noise term q_k is simply a Gaussian noise.

Experiment

The example illustrated in figure 10 is run with $\alpha = 0.5$, $T = 10$, $\Delta t = 0.1$ and 5 assimilations. The initial ensemble members are sampled from $N(0, 9)$. The model errors are sampled from $N(0, 1)$

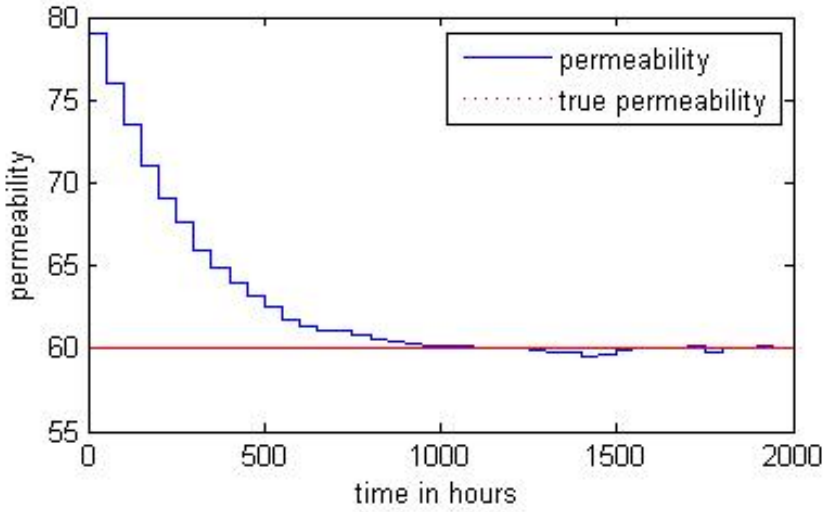

Figure 7. Permeability progress vs. true permeability.

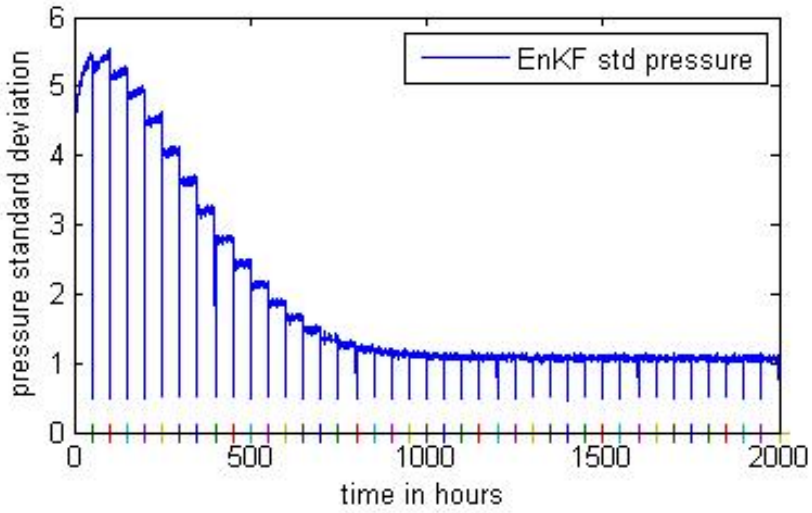

Figure 8. EnKF standard deviation for pressure.

and the measurements are sampled from $N(0, 0.5)$. The number of ensemble members is 1000, to eliminate the visual effects of using a finite ensemble.

4 Application of EnKF to history matching

Traditionally, history matching adjusts model parameters such as porosity, permeability, etc., such that the flow simulations using the adjusted parameters match observed production variables. This requires repeated flow simulations. In automatic history matching, we try to minimize the mismatch between measurements and computed values. This is often done by minimizing a so-called objective function. Some methods require computation of the gradient of the objective function, which can be time consuming when the number of parameters is large.

Several publications have discussed the use of Ensemble Kalman filtering to do history match-

ing. The method provides an approximate solution to the combined parameter and state estimation problem. The result is an ensemble of solutions approximating the posterior pdf for model parameters (porosity, permeability), state variables (pressure, saturation), and production data, conditioned to production history.

The EnKF methodology consists of two steps - the forecast step (stepping forward in time) and the assimilation step, where variables describing the state of the system are corrected to honor the observations. In history matching, the forward step is the flow simulation. It requires multiple, independent flow simulations (one for each ensemble member), but they can be performed simultaneously, and the method is ideal for parallel computing. It is not dependent on one specific type of reservoir simulator, it only requires the output from the simulator. The assimilation step consists of updating the model variables by using the Kalman equation.

Most of the traditional methods do not allow for continuous model updating, as new production data become available. The EnKF is able to do continuous updating of the reservoir model in the assimilation step. In this way, the reservoir models are always kept up-to-date.

4.1 Detailed description of EnKF for history matching

The EnKF can update both static (permeability and porosity) and dynamic (pressure, saturation) model parameters, as well as production data (production rates, bottom-hole pressures, gas-oil ratio etc.). The state vector x_i contains both model parameters (values for all grid cells) and production data. The filter is initialized by generating an initial ensemble X of the state vector, where the static model parameters are drawn from the prior distribution, conditioned to the model observations available at time t_0. The prior distribution is assumed to be Gaussian.

To take account for the model uncertainty, Gaussian noise is added to the ensemble members before each flow simulation. The observations are measurements of the production data in the wells. The observations must also be treated as random variables, in order to get a correct estimate of the variance in the ensemble. Therefore, random Gaussian noise ϵ is added to the observations, and new noise is generated for each ensemble member. The measurement errors are independent, so the covariance matrix R_e is diagonal. Because theoretical data are part of the state vector, the measurement operator H given in Equation 31 is a trivial matrix with only 0 and 1 as it components. We can arrange the matrix as

$$H = [\mathbf{0}|\mathbf{I}],$$

where $\mathbf{0}$ is a $N_d \times (N_y - N_d)$ matrix with zeroes and $\mathbf{I}$ is a $N_d \times N_d$ identity matrix. N_d is the number of observations and N_y is the total number of variables in the state vector. The number of observations might vary with time, and thus the matrix dimensions may change.

The static model parameters such as permeability and porosity are observed in wells before production starts. These observations are not updated.

The forecast step and the assimilation step are run sequentially. The flow simulations end at the next point in time where new observations are to be assimilated. The reservoir simulator is run once for each member of the ensemble. The forecasted state vector, resulting from applying the fluid flow simulator on the state vector from the last time step, is used to estimate the covariance matrix P_e, which again is used to find the Kalman gain matrix K. In the assimilation step, each member of the ensemble (the forecasted state vectors) is updated using the Kalman equation (equation 33). Note that in the forecast step (running the reservoir simulator), only dynamic variables are updated, and not the static variables such as permeability and porosity. The static variables are only updated in the assimilation step.

4.2 Investigations and improvements of the EnKF

The first application of Ensemble Kalman Filtering in history matching is described in Nævdal et al. (2002a) and Nævdal et al. (2002b). The filter was used to update static parameters in near-well reservoir models by tuning the permeability field. After this, several publications have demonstrated the use of EnKF for history matching, and suggested further developments of the method. In Nævdal et al. (2005), the filter is used to tune the permeability for simplified real-field reservoir simulation models. Gu and Oliver (2004) have tested EnKF on the PUNQ-S3 reservoir model.

Wen and Chen (2005) have investigated some important issues in the EnKF, for example the number of realizations in the ensemble. Their study showed that an ensemble size of 50 or 100 is too small to estimate the uncertainty. An ensemble size of 200 seems to be enough to represent the uncertainty in the model. Another issue is the covariance function. In a synthetic study, they investigated the sensitivity on the EnKF results, of using wrong covariance model. They found that the covariance model was not critically important in the EnKF. Wen and Chen (2005) also present a modified version of EnKF, where it is ensured that the updated static and dynamic variables are consistent. This will not always be the case, due to the fact that the Kalman updating is linear, while the flow equations are nonlinear. The modification consists of an extra flow simulation on the present time step, called the confirmation step. This flow simulation is run after the assimilation step, with the updated static parameters as input, and new dynamic parameters resulting from the confirmation step will replace the values from the assimilation step. In this way, the static and dynamic variables are consistent, and the flow simulation is then run for the next time step. The cost of this modification is a double in of the CPU time. Test on a synthetic reservoir shows that the confirmation step improves the match of production data.

Skjervheim et al. (2005) incorporate 4D seismic data in the history matching. They show how the EnKF can be used to update a combined reservoir simulation/seismic model using the combination of production data and inverted 4D seismic data. The method is completely recursive, with little additional cost compared to traditional EnKF. Test of the method on a synthetic case shows that adding seismic data gives a much better permeability estimate.

The ensemble of reservoir simulations provides a way to evaluate uncertainty in the reservoir description. If the ensemble of simulated reservoirs is obtained by correct sampling from the posterior probability density function, they give a characterization of the uncertainty in the reservoir model. Zafari and Reynolds (2005) discuss this. They provide examples where the performance of EnKF does not provide a reliable characterization of uncertainty, as well as examples where the method worked well.

The Gaussian assumption of EnKF is critical. In some multi-modal problems, EnKF can not sample the posterior pdf correctly, even if the number of ensembles is large. For two simple nonlinear toy problems, Zafari and Reynolds (2005) showed that the ensemble mean is not a good estimate of the true model. EnKF can not sample the posterior pdf correctly.

4.3 EnKF and Randomized Maximum Likelihood

Randomized maximum likelihood, Oliver et al. (1996), is a Bayesian method which is frequently used to generate approximate sampling of a pdf for a reservoir model conditioned to production data. The prior is assumed to be Gaussian. The idea is to find the model m that optimizes the following objective function:

$$\begin{aligned} S(m) &= (m-m_{us})' C_M^{-1} (m-m_{us}) \\ &\quad +[g(m)-(d_{obs}-C_D^{1/2}Z_D)]' C_D^{-1} \times [g(m)-(d_{obs}-C_D^{1/2}Z_D)] \end{aligned} \tag{41}$$

where C_M is the covariance of the random field, C_D is the variance of the measurements, m_{us}

is a sample from the prior, and $C_D^{1/2} Z_D$ is a vector of normally distributed errors with variance C_D.

In the linear case ($g(m)$ is a linear function), the method samples correctly from the posterior pdf. For the optimization, a Gauss-Newton method is used.

Zafari and Reynolds (2005) show that EnKF becomes equivalent with randomized maximum likelihood (RML) when the number of ensembles goes to infinity, provided that one begins with a prior multivariate Gaussian distribution for the model, data are uncorrelated in time and the relation between predicted data and the reservoir model is linear. In the linear case, RML provides correct sampling of the posterior pdf, and hence, EnKF will also do this when the ensemble size goes to infinity.

D.S. Oliver (EnKF workshop, Voss) proposes to combine randomized maximum likelihood and EnKF in a method called Ensemble Randomized Maximum likelihood. This is an iterative filter, and is more time consuming than the original EnKF.

4.4 Upscaling

Upscaling of reservoir properties is often required in order to perform the needed number of reservoir simulations. This results in bias in the production history conditioned reservoir representations when using the Ensemble Kalman filter. Lødøen and Omre (2005) show how to account for this when using the filter, and quantify loss in accuracy and precision. The idea is to use coarse scale fluid flow simulations results to predict fine scale fluid flow simulation results, and to assess the associated prediction uncertainty. The relation between the production properties from a coarse scale and a fine scale fluid flow simulator is estimated by the EnKF.

The up-scaling becomes an important issue when seismic data are taken into account, as these data usually appear on a much finer grid than the grid used for reservoir simulation.

4.5 Facies modeling

The pluri-Gaussian model is a pixel-based technique for facies modeling. The method is based on truncation of multiple Gaussian fields. Well conditioning is very difficult in the pluri-Gaussian model, and partly because of this, the model is not widely used. Liu and Oliver (2005) use the Ensemble Kalman filter to condition facies realizations generated by the pluri-Gaussian model to well data.

5 Concluding remarks

History matching of petroleum reservoirs is a difficult task, and is very CPU demanding because of the reservoir simulator, which must be run a large number of times. Use of the Ensemble Kalman filter for history matching allows for parallel reservoir simulator runs, which save time. The Ensemble Kalman filter is tested and further developed for history matching by several people, and the results are promising. But still, much work can be done in this field.

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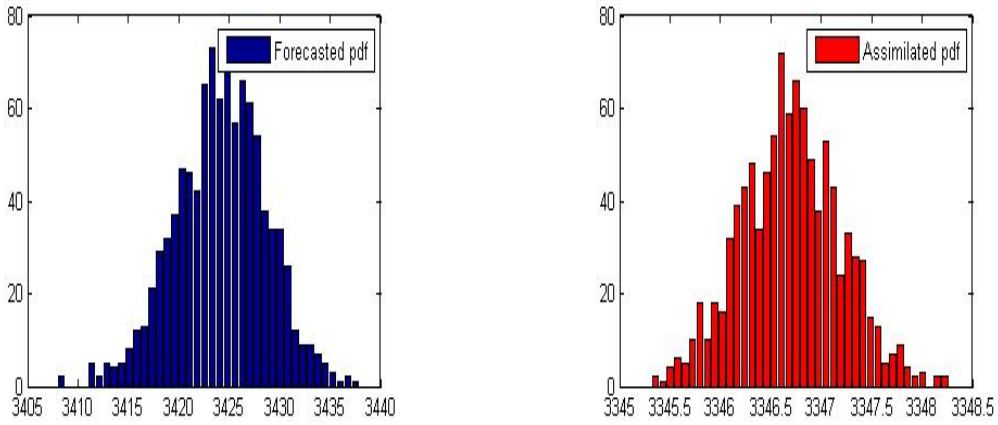

Figure 9. Histograms of ensembles before and after the assimilation step number 5.

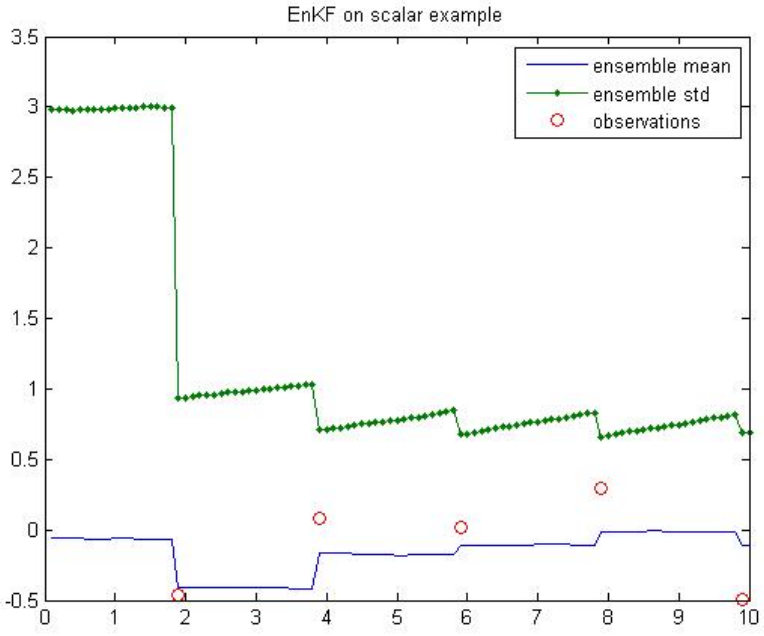

Figure 10. EnKF on scalar example from Evensen (2003).