

Tensors in Physics

Doing black holes and cosmology with Mathematica

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Tensors in Physics is a tensor component package published by the Scandinavian University Press. It comes with a book containing a brief course in differential geometry, a user's guide and reference manual for the *Mathematica* package CARTAN. CARTAN is an easy-to-use program for tensor component calculations. A large number of tensor expressions commonly used in gravitation theory and theoretical physics are predefined. The user-friendly high-level commands of CARTAN makes it an ideal tool for **interactive tensor calculations**.

In the demonstration it is shown how CARTAN can be used to verify the Schwarzschild solution of Einstein's field equations describing the gravitational field outside a static, spherical symmetric mass distribution. This is also the solution describing a static **black hole**.

Then I solve Einstein's field equations for a homogeneous, isotropic and spatially flat matter-dominated Universe. It is shown that the Kretschmann curvature invariant diverges at the origin of time. This is the **Big Bang**. Finally we check that the Kerr metric really solves Einstein's gravitational field equations in vacuum. This solution describes a rotating black hole. It took almost fifty years from the field equations were published until this solution was published. One can only guess how fast the solution had been found had Einstein had access to *Mathematica*.

Background

The metric tensor and the *vielbein*

$$\begin{aligned} ds^2 &= \sum_{\mu=1}^n \sum_{\nu=1}^n g_{\mu\nu} dx^\mu dx^\nu = \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{\mu=1}^n \sum_{\nu=1}^n \eta_{ij} e^i{}_\mu e^j{}_\nu dx^\mu dx^\nu \end{aligned}$$

Loading and initialization

```
In[1]:= << CARTAN.m
```

```
-----  
Tensors in Physics: CARTAN 1.2  
by Harald H. Soleng, October 23, 1996  
Copyright © 1996 by SCUP  
Scandinavian University Press  
Licensed to: Harald Soleng  
-----
```

Defining short hand notation

```
In[2]:= ∇x := CDerivative[x];  
Div[x_] := CDivergence[x, 1];  
Curl[x_] := Expand[Dual[tTimes[2, ∇x], {1, 2}]];
```

Warming up with vector analysis

Choosing a coordinate system

```
In[3]:= NewSession["polar.lib"]
```

```
In[4]:= SMetric
```

$$g_{rr} = 1$$

$$g_{\phi\phi} = r^2 \sin[\theta]^2$$

$$g_{\theta\theta} = r^2$$

Gradient of a scalar field

```
In[5]:= GradF = ∇F[r, ϕ, θ]
```

```
Out[5]= { { F(1,0,0)[r, ϕ, θ],  $\frac{\text{Csc}[\theta] F^{(0,1,0)}[r, \phi, \theta]}{r}$ ,  
           $\frac{F^{(0,0,1)}[r, \phi, \theta]}{r}$  } ,  
         { 1 } }
```

In[6]:= **GradFInCoordinateBasis = ToCoordinateBasis[GradF]**

Out[6]= $\{ \{ F^{(1,0,0)} [r, \phi, \theta], F^{(0,1,0)} [r, \phi, \theta], F^{(0,0,1)} [r, \phi, \theta] \}, \{ 1 \} \}$

Laplacian of scalar field

In[7]:= **LaplaceF = Expand[Div[GradF]]**

Out[7]=
$$\frac{\cot[\theta] F^{(0,0,1)} [r, \phi, \theta]}{r^2} + \frac{F^{(0,0,2)} [r, \phi, \theta]}{r^2} + \frac{\csc[\theta]^2 F^{(0,2,0)} [r, \phi, \theta]}{r^2} + \frac{2 F^{(1,0,0)} [r, \phi, \theta]}{r} + F^{(2,0,0)} [r, \phi, \theta]$$

Double covariant derivative of scalar field

In[8]:= **DDF = ∇ GradF**

Out[8]=

$$\left\{ \left\{ \left\{ F^{(2,0,0)} [r, \phi, \theta], \right. \right. \right.$$

$$\frac{1}{r^2} \left(\text{Csc}[\theta] \left(-F^{(0,1,0)} [r, \phi, \theta] + r F^{(1,1,0)} [r, \phi, \theta] \right) \right),$$

$$\left. \frac{-F^{(0,0,1)} [r, \phi, \theta] + r F^{(1,0,1)} [r, \phi, \theta]}{r^2} \right\},$$

$$\left\{ \frac{1}{r^2} \left(\text{Csc}[\theta] \left(-F^{(0,1,0)} [r, \phi, \theta] + r F^{(1,1,0)} [r, \phi, \theta] \right) \right), \right.$$

$$\frac{1}{r^2} \left(\text{Cot}[\theta] F^{(0,0,1)} [r, \phi, \theta] + \text{Csc}[\theta]^2 F^{(0,2,0)} [r, \phi, \theta] + r F^{(1,0,0)} [r, \phi, \theta] \right),$$

$$\left. -\frac{1}{r^2} \left(\text{Csc}[\theta] \left(\text{Cot}[\theta] F^{(0,1,0)} [r, \phi, \theta] - F^{(0,1,1)} [r, \phi, \theta] \right) \right) \right\},$$

$$\left\{ \frac{-F^{(0,0,1)} [r, \phi, \theta] + r F^{(1,0,1)} [r, \phi, \theta]}{r^2}, \right.$$

$$\left. -\frac{1}{r^2} \left(\text{Csc}[\theta] \left(\text{Cot}[\theta] F^{(0,1,0)} [r, \phi, \theta] - F^{(0,1,1)} [r, \phi, \theta] \right) \right), \right.$$

$$\left. \frac{F^{(0,0,2)} [r, \phi, \theta] + r F^{(1,0,0)} [r, \phi, \theta]}{r^2} \right\} \left. \right\},$$

$$\{1, 1\}$$

In[9]:= **Curl[GradF]**

Out[9]= $\{\{0, 0, 0\}, \{1\}\}$

Divergence and curl of vector field

In[10]:= **Vector = $\{\{V^r[r, \phi, \theta], V^\phi[r, \phi, \theta], V^\theta[r, \phi, \theta]\}, \{-1\}\}$**

Out[10]= $\{\{V^r[r, \phi, \theta], V^\phi[r, \phi, \theta], V^\theta[r, \phi, \theta]\}, \{-1\}\}$

In[11]:= **DivVector = Expand[Div[Vector]]**

Out[11]=
$$\begin{aligned} & \frac{2 V^r[r, \phi, \theta]}{r} + \\ & \frac{\cot[\theta] V^\theta[r, \phi, \theta]}{r} + \frac{(V^\theta)^{(0,0,1)}[r, \phi, \theta]}{r} + \\ & \frac{\csc[\theta] (V^\phi)^{(0,1,0)}[r, \phi, \theta]}{r} + \\ & (V^r)^{(1,0,0)}[r, \phi, \theta] \end{aligned}$$

In[12]:= **DerivativeOfVector = ∇ Vector**

Out[12]=

$$\left\{ \left\{ \left\{ (V^r)^{(1,0,0)} [r, \phi, \theta], -\frac{1}{r} (V^\phi [r, \phi, \theta] - \text{Csc}[\theta] (V^r)^{(0,1,0)} [r, \phi, \theta]), -\frac{V^\theta [r, \phi, \theta] - (V^r)^{(0,0,1)} [r, \phi, \theta]}{r} \right\}, \left\{ (V^\phi)^{(1,0,0)} [r, \phi, \theta], \frac{1}{r} (V^r [r, \phi, \theta] + \text{Cot}[\theta] V^\theta [r, \phi, \theta] + \text{Csc}[\theta] (V^\phi)^{(0,1,0)} [r, \phi, \theta]), \frac{(V^\phi)^{(0,0,1)} [r, \phi, \theta]}{r} \right\}, \left\{ (V^\theta)^{(1,0,0)} [r, \phi, \theta], -\frac{1}{r} (\text{Cot}[\theta] V^\phi [r, \phi, \theta] - \text{Csc}[\theta] (V^\theta)^{(0,1,0)} [r, \phi, \theta]), \frac{V^r [r, \phi, \theta] + (V^\theta)^{(0,0,1)} [r, \phi, \theta]}{r} \right\} \right\}, \{-1, 1\}$$

In[13]:= **CurlVector = Curl[Vector]**

Out[13]=
$$\left\{ \left\{ \frac{\cot[\theta] v^\phi[r, \phi, \theta]}{r} + \frac{(v^\phi)^{(0,0,1)}[r, \phi, \theta]}{r} - \frac{\csc[\theta] (v^\theta)^{(0,1,0)}[r, \phi, \theta]}{r}, \right. \right.$$
$$\frac{v^\theta[r, \phi, \theta]}{r} - \frac{(v^r)^{(0,0,1)}[r, \phi, \theta]}{r} + (v^\theta)^{(1,0,0)}[r, \phi, \theta], -\frac{v^\phi[r, \phi, \theta]}{r} + \frac{\csc[\theta] (v^r)^{(0,1,0)}[r, \phi, \theta]}{r} - (v^\phi)^{(1,0,0)}[r, \phi, \theta] \left. \right\},$$
$$\{1\}$$

In[14]:= **DivCurlVector = Div[CurlVector]**

Out[14]= 0

Verifying Schwarzschild's solution of Einstein's field equations

Let us first study Schwarzschild solution of Einstein's field equations. This is the geometry of a static black hole. The program comes with a library of geometries, and for simplicity we load the coordinate system and tetrad components from a file:

Loading the tetrad frame

```
In[15]:= NewSession["Schwarzs.lib"]
```

Computing the Riemann tensor

The system can now readily compute and show the non-zero components of the Riemann tensor. The results are shown in the user-defined rigid frame.

```
In[1]:= SRiemann
```

$$R_{1212} = -\frac{M}{r^3}$$

$$R_{1313} = -\frac{M}{r^3}$$

$$R_{1414} = -\frac{2M}{r^3}$$

$$R_{2323} = \frac{2M}{r^3}$$

$$R_{2424} = \frac{M}{r^3}$$

$$R_{3434} = \frac{M}{r^3}$$

Internally, the tensor is represented as a two-component list:

In[2]:= **Riemann**

Out[2]=

$$\left\{ \left\{ \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, -\frac{M}{r^3}, 0, 0 \right\}, \left\{ \frac{M}{r^3}, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, -\frac{M}{r^3}, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ \frac{M}{r^3}, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, 0, -\frac{2M}{r^3} \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ \frac{2M}{r^3}, 0, 0, 0 \right\} \right\} \right\}, \\ \left\{ \left\{ \left\{ 0, \frac{M}{r^3}, 0, 0 \right\}, \left\{ -\frac{M}{r^3}, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, \frac{2M}{r^3}, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, -\frac{2M}{r^3}, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, \frac{M}{r^3} \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, -\frac{M}{r^3}, 0, 0 \right\} \right\} \right\}, \\ \left\{ \left\{ \left\{ 0, 0, \frac{M}{r^3}, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \\ \left. \left. \left\{ -\frac{M}{r^3}, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \right. \\ \left. \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, -\frac{2M}{r^3}, 0 \right\}, \right. \right. \\ \left. \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\} \right\}$$

$$\begin{aligned}
& \left\{ 0, \frac{2M}{r^3}, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \\
& \quad \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \\
& \quad \left. \left\{ 0, 0, 0, \frac{M}{r^3} \right\}, \left\{ 0, 0, -\frac{M}{r^3}, 0 \right\} \right\}, \\
& \left\{ \left\{ \left\{ 0, 0, 0, \frac{2M}{r^3} \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \right. \\
& \quad \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ -\frac{2M}{r^3}, 0, 0, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, -\frac{M}{r^3} \right\}, \right. \\
& \quad \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, \frac{M}{r^3}, 0, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \\
& \quad \left. \left\{ 0, 0, 0, -\frac{M}{r^3} \right\}, \left\{ 0, 0, \frac{M}{r^3}, 0 \right\} \right\}, \\
& \left\{ \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\}, \right. \\
& \quad \left. \left\{ 0, 0, 0, 0 \right\}, \left\{ 0, 0, 0, 0 \right\} \right\} \left. \right\}, \\
& \left\{ 1, 1, 1, 1 \right\}
\end{aligned}$$

The first part is a list of tensor components, and the second part tells if the indices are **covariant** (marked by **1**) or **contravariant** (marked by **-1**):

In[3]:= **Riemann[[2]]**

Out[3]= {1, 1, 1, 1}

In[4]:= **Raise[Riemann, {1}][[2]]**

Out[4]= {-1, 1, 1, 1}

Computing the Ricci and Einstein tensors

When we ask for the Ricci tensor, we are told that the geometry is Ricci flat:

```
In[5]:= SRicci
```

```
This geometry is Ricci flat.
```

```
In[6]:= SEinstein
```

```
This is an Einstein space.
```

Curvature invariants

Computing invariants is very easy:

```
In[7]:= SKretschmann
```

$$R^{ijkl}R_{ijkl} = \frac{48 M^2}{r^6}$$

Taking the divergence of a rank–four tensor

Computing the covariant derivative or the divergence of a higher–rank tensor is usually a very time–consuming and boring task to do by hand. With CARTAN and Mathematica it is simple:

```
In[8]:= CDivergence[Riemann, 1]
```

```
Out[8]= {{{{0, 0, 0, 0}, {0, 0, 0, 0},  
           {0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0},  
           {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},  
         {{{0, 0, 0, 0}, {0, 0, 0, 0},  
           {0, 0, 0, 0}, {0, 0, 0, 0}}, {{0, 0, 0, 0},  
           {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},  
         {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},  
           {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},  
           {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}},  
         {1, 1, 1}}
```

Exercise in relativistic cosmology

Loading the geometry

```
In[9]:= NewSession["FLRW.lib"]
```

For simplicity, we assume that the Universe is **spatially flat**. Then

```
In[1]:= k = 0;
```

Checking the contracted Bianchi identity (conservation of energy)

According to Einstein's field equations, the Einstein tensor is proportional to the energy–momentum tensor. Since the Einstein tensor is divergence free:

```
In[2]:= Div[Einstein]
```

```
Out[2]= {{0, 0, 0, 0}, {1}}
```

the Energy–momentum tensor must also be covariantly conserved.

Finding Einstein's equations

```
In[3]:= SEinstein
```

$$G_{11} = - \frac{R' [t]^2 + 2 R [t] R'' [t]}{R [t]^2}$$

$$G_{22} = - \frac{R' [t]^2 + 2 R [t] R'' [t]}{R [t]^2}$$

$$G_{33} = - \frac{R' [t]^2 + 2 R [t] R'' [t]}{R [t]^2}$$

$$G_{44} = \frac{3 R' [t]^2}{R [t]^2}$$

The fourth component, corresponds to the time–time component of the energy–momentum tensor, i.e., the mass density. Assuming that ordinary galaxies dominate the mass of the Universe, the total mass scale as $1/R^3$, or

In[4]:= **Equation = Einstein[[1]][[4, 4]] == 8 Pi G K / R[t]^3**

Out[4]=
$$\frac{3 R'[t]^2}{R[t]^2} == \frac{8 G K \pi}{R[t]^3}$$

Solving with DSolve

In[5]:= **solution = DSolve[Equation, R[t], t]**

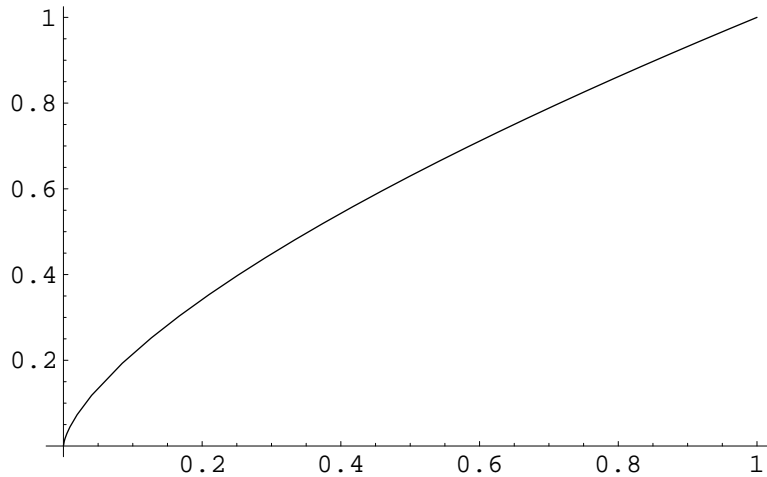
Out[5]=
$$\left\{ \left\{ R[t] \rightarrow -(-6)^{1/3} (G K \pi t^2 - 2 G K \pi t C[1] + G K \pi C[1]^2)^{1/3} \right\}, \right. \\ \left. \left\{ R[t] \rightarrow 6^{1/3} (G K \pi t^2 - 2 G K \pi t C[1] + G K \pi C[1]^2)^{1/3} \right\}, \right. \\ \left. \left\{ R[t] \rightarrow (-1)^{2/3} 6^{1/3} (G K \pi t^2 - 2 G K \pi t C[1] + G K \pi C[1]^2)^{1/3} \right\} \right\}$$

In[6]:= **solution = Simplify[solution[[2]] /. t -> t + C[1]]**

Out[6]=
$$\left\{ R[t + C[1]] \rightarrow (6 \pi)^{1/3} (G K t^2)^{1/3} \right\}$$

Identifying the Big Bang singularity

```
In[7]:= solution = {R[t] -> t^(2/3)};  
Plot[solution[[1]][[2]], {t, 0, 1}];
```



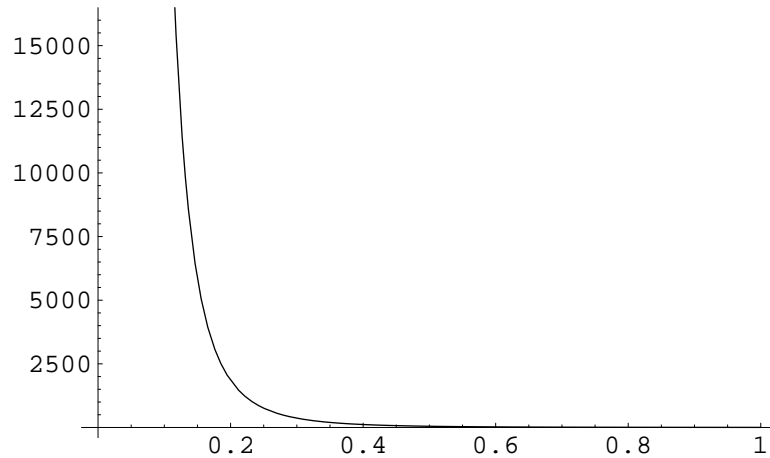
We substitute the solution back into the Riemann tensor and compute the Kretschmann invariant.

```
In[8]:= Riemann = Riemann /. DifferentialRules[solution];  
SKretschmann
```

$$R^{ijkl}R_{ijkl} = \frac{80}{27 t^4}$$

Clearly, there is a curvature singularity at $t=0$:


```
In[9]:= Plot[Kretschmann, {t, 0.1, 1}];
```



Verifying Kerr's solution

Loading the geometry

```
In[10]:= NewSession["Kerr.lib"]
```

Inspecting properties of the spacetime geometry

In[1]:= **SRiemann**

$$R_{1212} = \frac{4 M r (3 a^2 - 2 r^2 + 3 a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{1234} = - \frac{4 a M \cos[\theta] (a^2 - 6 r^2 + a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{1313} = \frac{4 M r (3 a^2 - 2 r^2 + 3 a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{1324} = \frac{4 a M \cos[\theta] (a^2 - 6 r^2 + a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{1414} = \frac{8 M r (3 a^2 - 2 r^2 + 3 a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{2323} = - \frac{8 M r (3 a^2 - 2 r^2 + 3 a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{2424} = - \frac{4 M r (3 a^2 - 2 r^2 + 3 a^2 \cos[2 \theta])}{(a^2 + 2 r^2 + a^2 \cos[2 \theta])^3}$$

$$R_{3434} = - \frac{4 M r (3 a^2 - 2 r^2 + 3 a^2 \cos [2 \theta])}{(a^2 + 2 r^2 + a^2 \cos [2 \theta])^3}$$

In[2]:= SKretschmann

$$R^{ijkl} R_{ijkl} = - (192 M^2 (a^2 - 2 r^2 + a^2 \cos [2 \theta]) (3 a^4 - 56 a^2 r^2 + 8 r^4 + 4 a^4 \cos [2 \theta] - 56 a^2 r^2 \cos [2 \theta] + a^4 \cos [4 \theta])) / (a^2 + 2 r^2 + a^2 \cos [2 \theta])^6$$

Verifying that Einstein's vacuum equation is satisfied

In[3]:= SRicci

This geometry is Ricci flat.

Implementational issues

Controlling swelling

A main problem when computing curvature tensors symbolically, is the **swelling** of the expressions. When manipulating tensors by differentiation and contraction, the intermediate expressions typically become very large and complicated. Usually, one can reduce the complexity significantly by a careful choice of simplification functions for the intermediate results. In CARTAN the swelling effect has been kept under control by **automatically invoking Factor or Expand**.

Installation and file handling

CARTAN is a large package with more than a hundred files and a large number of sub-packages that are loaded automatically when needed.

To install the package, it is necessary that the installation program is able to reconstruct a certain file structure. Unfortunately, **Mathematica expressions are not fully platform independent when it comes to file handling**. Therefore, it has been necessary to write **some platform dependent code in the installation section**.

```
In[4]:= FileNames["*", {"~/cartan"}]  
FileNames["*.m", {"~/cartan/program"}]
```

```
Out[4]= {~/cartan/allfiles.uu, ~/cartan/CARTAN.m,  
~/cartan/documnts, ~/cartan/extensns,  
~/cartan/initcart.m, ~/cartan/library,  
~/cartan/program, ~/cartan/READ.ME,  
~/cartan/startup}
```

Out[5]=

```
{~/cartan/program/antisym.m,  
~/cartan/program/cchristo.m,  
~/cartan/program/cconnect.m,  
~/cartan/program/ccottonc.m,  
~/cartan/program/ccotton.m,  
~/cartan/program/cehderiv.m,  
~/cartan/program/ceinhilb.m,  
~/cartan/program/ceinstc.m,  
~/cartan/program/ceinst.m,  
~/cartan/program/cfaraday.m,  
~/cartan/program/cgausbom.m,  
~/cartan/program/ch1tensr.m,  
~/cartan/program/ch2tensr.m,  
~/cartan/program/ch3tensr.m,  
~/cartan/program/chidden.m,  
~/cartan/program/chiralga.m,  
~/cartan/program/chtensor.m,  
~/cartan/program/ckretsc.m,  
~/cartan/program/ckretsch.m,  
~/cartan/program/clanczos.m,  
~/cartan/program/clcartan.m,  
~/cartan/program/clinout.m,  
~/cartan/program/clmemory.m,  
~/cartan/program/cmaxwell.m,  
~/cartan/program/cmetric.m,  
~/cartan/program/cnpcurva.m,  
~/cartan/program/cricci2c.m,  
~/cartan/program/cricci2.m,  
~/cartan/program/criccic.m,  
~/cartan/program/cricci.m,  
~/cartan/program/criemanc.m,  
~/cartan/program/criemann.m,  
~/cartan/program/csgenera.m,  
~/cartan/program/cspincoe.m,  
~/cartan/program/cstruct.m,  
~/cartan/program/ctijmaxw.m,  
~/cartan/program/ctorsion.m,  
~/cartan/program/cumetric.m,  
~/cartan/program/cweylc.m,  
~/cartan/program/cweylinc.m,  
~/cartan/program/cweylinv.m,  
~/cartan/program/cweyl.m,
```

~/cartan/program/declarat.m,
~/cartan/program/definiti.m,
~/cartan/program/diracgam.m,
~/cartan/program/headcart.m,
~/cartan/program/helpcart.m,
~/cartan/program/helpcomp.m,
~/cartan/program/helpcont.m,
~/cartan/program/helpconv.m,
~/cartan/program/helpfunc.m,
~/cartan/program/helpshow.m,
~/cartan/program/helpvari.m,
~/cartan/program/inputcar.m,
~/cartan/program/inputffi.m,
~/cartan/program/inputint.m,
~/cartan/program/inputpro.m,
~/cartan/program/inputses.m,
~/cartan/program/lecifive.m,
~/cartan/program/lecisix.m,
~/cartan/program/license.m,
~/cartan/program/majorana.m,
~/cartan/program/maxwella.m,
~/cartan/program/namesdec.m,
~/cartan/program/namestor.m,
~/cartan/program/npftensr.m,
~/cartan/program/perturb.m,
~/cartan/program/progrdta.m,
~/cartan/program/savecart.m,
~/cartan/program/scconnec.m,
~/cartan/program/sccontor.m,
~/cartan/program/sccottoc.m,
~/cartan/program/sccotton.m,
~/cartan/program/scctensr.m,
~/cartan/program/sceinstc.m,
~/cartan/program/sceinst.m,
~/cartan/program/scfarada.m,
~/cartan/program/schltens.m,
~/cartan/program/sch2tens.m,
~/cartan/program/sch3tens.m,
~/cartan/program/schr isto.m,
~/cartan/program/schtensr.m,
~/cartan/program/sclanczo.m,
~/cartan/program/scmaxwel.m,
~/cartan/program/sconnecc.m,

~/cartan/program/sconnect.m,
~/cartan/program/scontort.m,
~/cartan/program/scottonc.m,
~/cartan/program/scotton.m,
~/cartan/program/scriccic.m,
~/cartan/program/scricci.m,
~/cartan/program/scriemac.m,
~/cartan/program/scrieman.m,
~/cartan/program/sctijmax.m,
~/cartan/program/sctorsio.m,
~/cartan/program/scweylc.m,
~/cartan/program/scweyl.m,
~/cartan/program/sehderiv.m,
~/cartan/program/seinhilb.m,
~/cartan/program/seinstc.m,
~/cartan/program/seinst.m,
~/cartan/program/sfaraday.m,
~/cartan/program/sgausbon.m,
~/cartan/program/sh1tensr.m,
~/cartan/program/sh2tensr.m,
~/cartan/program/sh3tensr.m,
~/cartan/program/shtensor.m,
~/cartan/program/sinvaric.m,
~/cartan/program/sinvari.m,
~/cartan/program/skretsc.c.m,
~/cartan/program/skretsch.m,
~/cartan/program/slanczos.m,
~/cartan/program/smaxwell.m,
~/cartan/program/smetric.m,
~/cartan/program/snpcurva.m,
~/cartan/program/spinor.m,
~/cartan/program/sricci2c.m,
~/cartan/program/sricci2.m,
~/cartan/program/sriccic.m,
~/cartan/program/sricci.m,
~/cartan/program/sriemanc.m,
~/cartan/program/sriemann.m,
~/cartan/program/sscurvac.m,
~/cartan/program/sscurvat.m,
~/cartan/program/sspincoe.m,
~/cartan/program/sstruct.m,
~/cartan/program/stijmaxw.m,
~/cartan/program/storsion.m,

```
~/cartan/program/sumetric.m,  
~/cartan/program/sweylc.m,  
~/cartan/program/sweylinc.m,  
~/cartan/program/sweylinv.m,  
~/cartan/program/sweyl.m,  
~/cartan/program/transcri.m,  
~/cartan/program/transinp.m,  
~/cartan/program/wannasee.m,  
~/cartan/program/wrkngdir.m}
```

How to get CARTAN? Surf the net

<http://store.wolfram.com/view/cartan/>

<http://www.scup.no/books/en/cartan/>