ON THE APPROXIMATION OF THE SOLUTION OF THE PRESSURE EQUATION BY CHANGING THE DOMAIN

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Abstract. The purpose of this paper is to study a well-known technique for simplifying the pressure and velocity computations in models arising in reservoir simulation and metal casting. In both cases the domain of the pressure equation is changed. The equations considered are self-adjoint second order elliptic problems with coefficient functions representing the permeability of the porous media of concern. In the domain modification procedures, low permeability areas are either removed or inserted in order to simplify the computations. Using the theory of Sobolev spaces, we prove that the approximations converge toward the correct solution as the permeability tends to zero in the proper regions. Finally, we present a numerical experiment which indicates that the estimated rate of convergence is sharp.

Key words. reservoir simulation, metal casting, elliptic problems, domain modifications.

AMS(MOS) subject classifications. 35B99, 35J25, 65N22.

1. Introduction. It is generally accepted that mathematical models of fluid flow in porous media may be stated in terms of coupled partial differential equations, see for instance Ewing [10], Peaceman [20] or Ni and Beckermann [19] and references therein. Usually, a pressure equation is derived by appealing to Darcy's law and conservation of mass. In this paper we will concentrate on the elliptic equation

$$\nabla \cdot [\Lambda (\nabla p - \mathbf{Q})] + f = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.1}$$

which may be taken as a prototype of pressure equations arising both in models of oil recovery and metal casting. In (1.1) p represents the unknown fluid pressure and Λ is a second order mobility tensor incorporating various physical parameters such as the permeability of the medium and the viscosity of the fluid. Furthermore, $\mathbf{Q}$ and f are given functions representing various physical data. A more detailed description of our prototype model and the parameters involved will be given in the next section.

Oil reservoirs will frequently contain low permeable zones. In these zones the

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mobility Λ can become arbitrary close to zero. In such cases, the stiffness matrix obtained from a finite element approximation of (1.1) is badly conditioned. This is a well known problem, and causes serious problems for the efficient numerical solution of (1.1). To overcome this problem it is customary to ignore areas of very low mobility, by putting $\Lambda = 0$ and solve the problem on the remaining part of Ω, i.e. the part where Λ is significantly greater than zero. Hence, a problem close to degeneracy is approximated with a more well behaved problem. The basic idea of this procedure is that very low permeability zones do not contribute significantly to the overall flow picture and may thus be eliminated. This approximation is used in state-of-the-art simulators, see e.g. [21] and Peaceman [20], but to our knowledge, the errors introduced by this procedure have not been analyzed. It is the purpose of this paper to prove that the method converges, in the sense that the error goes to zero as the permeability in the removed region goes to zero.

A problem of the same flavour arises in the modelling of metal casting. Consider the solidification of some metal which is initially completely in the liquid phase. As the process goes on, a mushy region arises in the melt. In this region, Darcy's law applies and the velocity field can be computed from a pressure equation of the form (1.1), cf. e.g. Haug, Mo and Thevik [14]. As the process goes further, small areas of solid phase metal appear in the mushy region. In these areas the rate of flow is zero and they should be removed from the domain of the pressure equation. Obviously, removing such areas imply a tremendous grid generation task; for each time-step in the overall simulation, an increasingly complicated domain must be re-grided. A practical approach to this problem is simply to allow the areas of zero permeability, i.e. the area where the metal is in solid phase, to remain in the solution domain and set the permeability in this area equal to a very small constant. Obviously, this procedure generate badly conditioned finite element discretizations as discussed above, but this is considered to be less cumbersome than having to re-grid the entire domain in each time-step.

Models of the form (1.1) arise in a series of applications. Basic mathematical properties and numerical methods are discussed in the books of Hackbusch [13], Marti [17], cf. also Ciarlet [6], Dautray and Lions [7] and Gilbarg and Trudinger [11]. A method for changing the domain of the Helmholtz equation is discussed by Keller [15]. So-called fictitious domain methods for preconditioning of elliptic problems are discussed by several authors, see for instance Börgers and Widlund [4] or Glowinski, Pan and Périaux [12] and references therein. Stability of (1.1) with respect to the mobility tensor Λ is discussed by Bruaset and Nielsen [5].

Our aim in this paper is to show that both the simplification procedure used in reservoir simulation and the one used in metal casting work well. More precisely, we will prove that the solutions of the simplified problems converge toward the correct solutions as the permeability parameter tends to zero in the proper areas.

The rest of this paper is organized as follows: In the next section we describe a generic model problem covering both applications of interest. Thereafter we present our main results. Section 3 contains the necessary mathematical preliminaries and Section 4 contains the proofs of the convergence results. In Section 5 we derive the convergence results for the discrete case and supplement our theoretical results by a numerical experiment. Finally, we make some concluding remarks in Section 6 .

2. The model problems. As discussed above, the problem of approximating the solution of the pressure equation by changing the domain arises in several applications. We focus on two cases: oil recovery and metal casting, but clearly problems of this type will appear in a lot of models. We have not been able to find any rigorous analysis of the method in the literature, although it is commonly used at least in reservoir simulation.

Since this technique appears in different applications, we will try to consider a somewhat generic model problem. To this end, we start by observing that the pressure equations arising in models of metal casting and oil-recovery, can be written in a common form. The pressure equation for oil reservoir models is usually written in the following form

$$\nabla \cdot \left[\Lambda_\epsilon \left(\nabla p_\epsilon - \rho g \nabla D\right)\right] + \frac{q}{\rho} = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.1}$$

see for instance Peaceman [20]. As mentioned above, p_ϵ represents the unknown fluid pressure and Λ_ϵ is a second order mobility tensor incorporating physical parameters such as the permeability of the medium and the viscosity of the fluid. Moreover, the function D denotes the depth of the reservoir measured in the direction of gravity, while g is the gravitational constant and ρ is the fluid density. Depending on the exact definition of Λ_ϵ, (2.1) may be taken as a prototype of the pressure equations for single-phase as well as multi-phase flows. For heterogeneous reservoirs, the mobility may have large variations and even discontinuities. Typically, Λ_ϵ can be piecewise constant, thus representing the effect of different reservoir layers. The function q in (2.1) represents internal sources. Finally, the subscript ϵ indicates the size of the mobility in the low permeable zones of the reservoir. The precise use of this subscript is explained below.

Similarly, the pressure equation arising in models of mush regions in metal solidification can be written on the form (cf. Haug, Mo and Thevik [14] and Ni and Beckerman [19]),

$$\nabla \cdot \left[\Lambda_\epsilon \left(\nabla p_\epsilon + (0, \frac{\rho g H_s}{p_0 + \rho g H_s}) \right) \right] = k \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.2}$$

where again p_ϵ denotes the unknown pressure, Λ_ϵ is a second order mobility tensor, ρ is the density of the melt and g is the gravitational constant. Furthermore, H_s represents the depth of the melt in direction of gravity, while p_0 is the atmospheric pressure. Moreover, the function k is essentially derived from the volume fraction of the liquid phase in the melt. Finally, the subscript ϵ denotes the order of artificial mobility introduced in the solid zones of the mushy region. For further details on mathematical models of metal casting we refer to Haug, Mo and Thevik [14] and references therein.

In both models (2.1) and (2.2), we assume that the domain Ω can be divided into two connected and open domains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \bigcup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Furthermore, Ω_1 and Ω_2 are chosen such that the mobility is very low in Ω_2 and $O(1)$ in Ω_1. To clarify this point we let $\Lambda = \Lambda(x)$ be a $O(1)$ mobility tensor, and we assume that Ω is constructed such that

$$\Lambda_\epsilon(x) = \begin{cases} \Lambda(x) & \text{for } x \in \Omega_1 \\ \epsilon\Lambda(x) & \text{for } x \in \Omega_2, \end{cases} \tag{2.3}$$

where $0 < \epsilon \ll 1$ and $\Lambda(x) = (\lambda_{i,j}(x))$ is a symmetric matrix with entries $\lambda_{i,j} : \Omega \to \mathbb{R}$. Thus we have gathered the areas of low mobility in Ω_2, whereas Ω_1 denotes the area of $O(1)$ mobility. A prototypical domain of a problem of this type is shown in Figure 1. The precise assumption on $\Lambda = \Lambda(x)$ will be stated below.

With this notation at hand, we note that both equations (2.1) and (2.2) can be written in the form

$$\nabla \cdot [\Lambda_\epsilon (\nabla p_\epsilon - \mathbf{Q})] + f = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.4}$$

where $\mathbf{Q} : \Omega \to \mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$ are suitable defined functions. In the mathematical model of metal casting in (2.2), $f = -k$, where k is a function obtained from derivatives of the volume fraction of the liquid phase in the melt. In this model, Ω_2 represents an area of solid phase metal, and of course the volume fraction of the liquid phase is equal to zero in this subdomain. Hence, $k(x) = 0$ for all $x \in \Omega_2$.

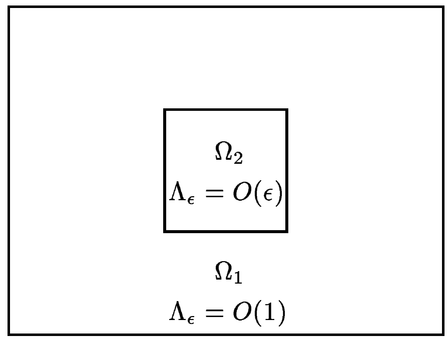

FIG. 1. *An example of a solution domain* $\Omega = \Omega_1 \cup \overline{\Omega}_2$.

Clearly, for equation (2.1), $f = q/\rho$, where, as mentioned above, q represents internal sources and ρ is the density of the fluid in question. One would not expect any wells in Ω_2, since the mobility is low in this zone, i.e. $q = 0$ in Ω_2. Based on these observations, we find it natural to assume that

$$f|_{\Omega_2} = 0 \tag{2.5}$$

throughout this paper. Equation (2.4) is our prototype of the pressure equation.

The boundary $\partial\Omega$, which is assumed to be sufficiently smooth, can be divided into two disjoint segments Γ_{neu} and Γ_{dir}. The pressure equation (2.4) is then subject to the boundary conditions

$$\begin{aligned} \mathbf{v}_\epsilon \cdot \mathbf{n} &= g_{\text{neu}} \qquad \text{on } \Gamma_{\text{neu}}, \\ p_\epsilon &= p_{\text{dir}} \qquad \text{on } \Gamma_{\text{dir}}. \end{aligned} \tag{2.6}$$

Here $\mathbf{n}$ denotes the outwards directed normal vector of unit length, and g_{neu} and p_{dir} are given functions defined on Γ_{neu} and Γ_{dir}, respectively. . The Darcy velocity $\mathbf{v}_\epsilon$ is defined as

$$\mathbf{v}_\epsilon = -\Lambda_\epsilon \left(\nabla p_\epsilon - \mathbf{Q}\right). \tag{2.7}$$

The approximation of p_ϵ that we want to study is the following: Remove Ω_2 from the domain, and introduce a Neuman type boundary condition on $\partial\Omega_2$. More precisely, we let p be the solution of

$$\nabla \cdot [\Lambda (\nabla p - \mathbf{Q})] + f = 0 \quad \text{in } \Omega_1, \tag{2.8}$$

with boundary conditions similar to (2.6) on $\partial\Omega$ and noflow on $\partial\Omega_2$, i.e.[1]

$$\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= g_{\text{neu}} && \text{on } \Gamma_{\text{neu}}, \\ p &= p_{\text{dir}} && \text{on } \Gamma_{\text{dir}}, \\ \mathbf{v} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega_2. \end{aligned} \tag{2.9}$$

Clearly, in this problem the pressure p is only defined on Ω_1. Hence, the Darcy velocity will also be defined on Ω_1. Furthermore, since there is no flow through $\partial\Omega_2$, it is reasonable to extend the velocity field to the whole domain Ω by putting

$$\mathbf{v} = \begin{cases} -\Lambda (\nabla p - \mathbf{Q}) & \text{on } \Omega_1, \\ 0 & \text{on } \Omega_2. \end{cases} \tag{2.10}$$

It should be noted that this extension of the velocity field is frequently applied in practical computations, where the computed velocity is used as input to other equations in the overall simulation process.

The rest of this paper is concerned with the following two problems: How well does p approximate p_ϵ?, and how well does $\mathbf{v}$ approximate $\mathbf{v}_\epsilon$? Our main results are stated in the following two theorems.

THEOREM 2.1. *Let p_ϵ and p be the weak solutions of (2.4), (2.6) and (2.8), (2.9), respectively. Suppose f satisfies (2.5), then there exists a constant c, independent of ϵ, such that*

$$\|p_\epsilon - p\|_{H^1(\Omega_1)} \leq c\epsilon.$$

THEOREM 2.2. *Let p_ϵ and p be the weak solutions of (2.4), (2.6) and (2.8), (2.9),respectively. Furthermore, assume that f satisfies (2.5). Then the difference*

[1] Obviously, in the case of models of metal casting, this is the original mathematical model and (2.2) it's approximation. Recall that Ω_2 represents an area of solid phase metal where the rate of flow is equal to zero. From a mathematical point of view, these matters are not important, and we will always refer to p as the approximation of p_ϵ.

between the velocity vectors $\mathbf{v}_\epsilon$ *and* $\mathbf{v}$ *defined in (2.7) and (2.10) satisfy*

$$\| \mathbf{v}_\epsilon - \mathbf{v} \|_{(L^2(\Omega))^2} \le c\epsilon,$$

where c *is a constant independent of* ϵ.

Here $\| \cdot \|_{H^1(\Omega_1)}$ denotes the usual H^1-Sobolev norm on Ω_1, whereas $\| \cdot \|_{(L^2(\Omega))^2}$ is defined by

$$\| \mathbf{w} \|^2_{(L^2(\Omega))^2} = \| w_1 \|^2_{L^2(\Omega)} + \| w_2 \|^2_{L^2(\Omega)} = \int_\Omega |\mathbf{w}|^2 \, dx,$$

for $\mathbf{w} = (w_1, w_2)^T \in (L^2(\Omega))^2$.

Remarks.

1. In this paper we assume that there is only one subdomain Ω_2 in the domain Ω where the mobility is of order $O(\epsilon)$. Of course, in real-world simulations there can be a number of such subdomains. The analysis presented in the present paper can be extend to the case of a finite number of subdomains with $O(\epsilon)$ mobility.
2. In the present paper we consider two dimensional models. However, it should be noted that similar results hold in the case of three space dimensions.
3. Discrete versions of the results above will be given in Section 5.

Until the numerical experiment presented in Section 5 we will be concerned with proving Theorems 2.1 and 2.2.

3. Preliminaries and weak formulations. In this paper we use $L^p(Y)$, for $Y = \Omega, \Omega_1, \Omega_2, \Gamma_{\text{neu}}$, to denote the classical L^p spaces of real valued functions defined on Y. For $\mathbf{w} \in \mathbb{R}^2$, $|\mathbf{w}|$ denotes the Euclidean norm of $\mathbf{w}$. The Sobolev spaces $H^1(\Omega)$, $H^1(\Omega_1)$ and $H^1(\Omega_2)$ are as usual defined by

$$H^1(Y) = \left\{ \psi \in L^2(Y);\ \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \in L^2(Y) \right\} \quad \text{for } Y = \Omega, \Omega_1, \Omega_2.$$

Here $\partial\psi/\partial x$ and $\partial\psi/\partial y$ are the distributional partial derivatives of ψ. The appropriate subspaces for our model problems, due to the boundary conditions (2.6) and (2.9), are

$$\begin{aligned} V_\Omega &= \left\{ \psi \in H^1(\Omega);\ \psi = 0 \text{ on } \Gamma_{\text{dir}} \right\}, \\ V_{\Omega_1} &= \left\{ \psi \in H^1(\Omega_1);\ \psi = 0 \text{ on } \Gamma_{\text{dir}} \right\}. \end{aligned}$$

In order to get well-posed variational problems, which will be presented below, we will assume that

$$\mathbf{Q} \in L^2(\Omega) \times L^2(\Omega), \ \ g_{\text{neu}} \in L^2(\Gamma_{\text{neu}}), \ \ f \in L^2(\Omega), \text{ and } p_{\text{dir}} \in H^{1/2}(\Gamma_{\text{dir}}). \tag{3.1}$$

Based on the discussion of the model problems in Section 2, the function f satisfies $f|_{\Omega_2} = 0$. Furthermore, we will assume that $\Lambda(x) = (\lambda_{i,j}(x))$ is a symmetric uniformly positive definite matrix with entries in $L^\infty(\Omega)$. More precisely, there are finite constants m and M, independent of ϵ and x, such that

$$0 < m \le \frac{\mathbf{z}^T \Lambda(x) \mathbf{z}}{|\mathbf{z}|^2} \le M \quad \text{for all } \mathbf{z} \in \mathbb{R}^2 \setminus \{0\} \text{ and } x \in \Omega. \tag{3.2}$$

Thus, our equations are within the class of strictly or strongly elliptic problems, see for instance Dautray and Lions [8, Ch. II.8]. For $x \in \Omega$ the operator norm $|\Lambda(x)|$ is, as usual, defined by

$$|\Lambda(x)| = \sup_{\mathbf{z} \in \mathbb{R}^2 \setminus \{0\}} \frac{|\Lambda(x)\mathbf{z}|}{|\mathbf{z}|}.$$

From (3.2) it follows that Λ satisfies the inequality

$$0 < m \le |\Lambda(x)| \le M \quad \text{for all } x \in \Omega, \tag{3.3}$$

and then from the definition (2.3) of Λ_ϵ, we get

$$\begin{aligned} 0 < m \le |\Lambda_\epsilon(x)| \le M \quad & \text{for all } x \in \Omega_1, \\ 0 < \epsilon m \le |\Lambda_\epsilon(x)| \le \epsilon M \quad & \text{for all } x \in \Omega_2. \end{aligned} \tag{3.4}$$

Next, we will assume that the subdomains Ω_1 and Ω_2 have sufficiently smooth boundaries and that $\Omega = \Omega_1 \bigcup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\partial\Omega \cap \overline{\Omega}_2 = \emptyset$. That is, $\overline{\Omega}_2$ is contained in Ω and $\text{dist}(\overline{\Omega}_2, \partial\Omega) > 0$. Moreover, it should be noted that $\partial\Omega \subset \partial\Omega_1$, $\partial\Omega_2 \subset \partial\Omega_1$ and $\partial\Omega \bigcup \partial\Omega_2 = \partial\Omega_1$.

Assumption (3.1) and the trace theorem implies that there exists an extension $\overline{p}_{\text{dir}} \in H^1(\Omega)$ of p_{dir} such that $T(\overline{p}_{\text{dir}})|_{\Gamma_{\text{dir}}} = p_{\text{dir}}$. Here, $T : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the trace operator with operator norm

$$\|T\| = \sup_{\psi \in H^1(\Omega) \setminus \{0\}} \frac{\|T(\psi)\|_{H^{1/2}(\partial\Omega)}}{\|\psi\|_{H^1(\Omega)}}.$$

Let $\overline{p}_{\mathrm{dir},\Omega_1} = \overline{p}_{\mathrm{dir}}|_{\Omega_1}$ then we can introduce the sets

$$\begin{aligned} \overline{p}_{\mathrm{dir}} + V_\Omega &= \{\overline{p}_{\mathrm{dir}} + u;\ u \in V_\Omega\}, \\ \overline{p}_{\mathrm{dir},\Omega_1} + V_{\Omega_1} &= \{\overline{p}_{\mathrm{dir},\Omega_1} + u;\ u \in V_{\Omega_1}\}. \end{aligned}$$

Now, the weak formulation of (2.4), (2.6) can be defined as follows: Find $p_\epsilon \in \overline{p}_{\mathrm{dir}} + V_\Omega$ such that

$$\int_\Omega \nabla\psi \cdot [\Lambda_\epsilon(\nabla p_\epsilon - \mathbf{Q})]\, dx = \int_{\Omega_1} f\psi\, dx - \int_{\Gamma_{\mathrm{neu}}} \psi g_{\mathrm{neu}}\, ds \tag{3.5}$$

for all $\psi \in V_\Omega$. It should be noted that here we have used the fact that $f|_{\Omega_2} = 0$, which explains the integral over Ω_1 in (3.5). Next, the weak formulation of (2.8), (2.9) is: Find $p \in \overline{p}_{\mathrm{dir},\Omega_1} + V_{\Omega_1}$ such that

$$\int_{\Omega_1} \nabla\varphi \cdot [\Lambda(\nabla p - \mathbf{Q})]\, dx = \int_{\Omega_1} f\varphi\, dx - \int_{\Gamma_{\mathrm{neu}}} \varphi g_{\mathrm{neu}}\, ds \tag{3.6}$$

for all $\varphi \in V_{\Omega_1}$.

It follows from the Lax-Milgram theorem (see for instance Dautray and Lions [7] or Gilbarg and Trudinger [11]), that if $\mathbf{Q}$, g_{neu}, f and p_{dir} satisfy (3.1) and Λ satisfies (3.3), then the problem (3.5) has a unique solution $p_\epsilon \in \overline{p}_{\mathrm{dir}} + V_\Omega$ for all $\epsilon \in (0,1)$, and the problem (3.6) has a unique solution $p \in \overline{p}_{\mathrm{dir},\Omega_1} + V_{\Omega_1}$.

In both the applications we mentioned above a velocity field is defined by differentiating the pressure. Since $p_\epsilon \in H^1(\Omega)$, $p \in H^1(\Omega_1)$, $\mathbf{Q} \in (L^2(\Omega))^2$ and Λ is assumed to satisfy (3.2), it follows that the velocity vectors, defined by

$$\begin{aligned} \mathbf{v}_\epsilon &= -\Lambda_\epsilon(\nabla p_\epsilon - \mathbf{Q}) \quad \text{on } \Omega, \\ \mathbf{v} &= \begin{cases} -\Lambda\,(\nabla p - \mathbf{Q}) & \text{on } \Omega_1, \\ 0 & \text{on } \Omega_2, \end{cases} \end{aligned}$$

belong to $(L^2(\Omega))^2$. We are primarily interested in obtaining a bound on the difference $\mathbf{v} - \mathbf{v}_\epsilon$ since the velocity field is used as input to other equations both in reservoir simulation and simulation of metal casting. Actually, in both cases, the pressure is not an important quantity: it is the derivatives of the pressure that is used. To measure the difference between the velocity vectors we will use the $\|\cdot\|_{(L^2(\Omega))^2}$-norm defined at the end of the previous section.

For easy reference, we close this section by stating Poincaré's inequality for V_{Ω_1}:

There is a constant c_1, only depending on the domain Ω_1, such that

$$\|\varphi\|_{L^2(\Omega_1)} \leq c_1 \left(\int_{\Omega_1} |\nabla \varphi|^2 \right)^{1/2} \quad \forall \varphi \in V_{\Omega_1}, \tag{3.7}$$

see for instance Dautray and Lions [7, Ch. IV.7].

Now we have made the necessary preparations to prove Theorems 2.1 and 2.2.

4. The convergence results. The purpose of this section is to prove that the solution and the associated velocity field of (3.5) converges towards the solution and the associated velocity field of (3.6) as ϵ goes to zero. The results have already been stated in Theorems 2.1 and 2.2 in Section 2.

Before we present the proofs of Theorem 2.1 and 2.2, we shall briefly discuss why these results are obtainable. Recall that p_ϵ is the solution of (3.5). For this problem, a straightforward application of the Lax-Milgram Theorem leads to a bound of the form $\|p_\epsilon\|_{H^1(\Omega)} \leq O(1/\epsilon)$. Thus indicating that some sort of problem will arise as ϵ tends to zero. However, we shall prove below that this bound is very pessimistic when condition (2.5) is fulfilled. In this case, we will show that $\|p_\epsilon\|_{H^1(\Omega)} = O(1)$, thus bounded independent of ϵ, and we obtain a well defined limit. The limit of p_ϵ on Ω_1 turns out to be defined by solving the problem on Ω_1 with a Neuman boundary condition on $\partial\Omega_2$.

This procedure is very simple in 1D, and we therefore present a trivial but illustrative example. Let $p_\epsilon = p_\epsilon(x)$, $x \in [0, 3]$ be the weak solution of the following two-point boundary value problem

$$\begin{aligned} (k(x)p_\epsilon'(x))' &= f(x) \quad \text{for } 0 < x < 3, \\ p_\epsilon(0) = 0 \quad &\text{and} \quad p_\epsilon(3) = 1. \end{aligned} \tag{4.1}$$

Here

$$(f(x), k(x)) = \begin{cases} (-1, 1) & \text{for } 0 \leq x \leq 1 \\ (0, \epsilon) & \text{for } 1 < x < 2 \\ (-1, 1) & \text{for } 2 \leq x \leq 3, \end{cases}$$

where $\epsilon > 0$ is a constant. Obviously, (4.1) is a 1D version of the 2D problem discussed above. The solution of this problem is given by

$$p_\epsilon(x) = \begin{cases} \frac{3\epsilon+1}{2\epsilon+1}x - \frac{1}{2}x^2 & \text{for } 0 \le x \le 1, \\ \left(\frac{3\epsilon+1}{2\epsilon+1} - \frac{1}{2}\right) + \frac{1}{2\epsilon+1}(x-1) & \text{for } 1 < x < 2, \\ -\left(\frac{3\epsilon}{2\epsilon+1} + \frac{1}{2}\right) + \frac{5\epsilon+2}{2\epsilon+1}x - \frac{1}{2}x^2 & \text{for } 2 \le x \le 3. \end{cases}$$

For this problem, we easily observe that p_ϵ is well-behaved as ϵ tends to zero. We note that $\| p_\epsilon \|_{H^1(0,3)} = O(1)$, and that $p = \lim_{\epsilon \to 0} p_\epsilon$ is given by

$$p(x) = \begin{cases} x - \frac{1}{2}x^2 & \text{for } 0 \le x \le 1, \\ x - \frac{1}{2} & \text{for } 1 < x < 2, \\ -\frac{1}{2} + 2x - \frac{1}{2}x^2 & \text{for } 2 \le x \le 3. \end{cases}$$

Now this latter function p is also the solution of the problem defined by

$$\begin{aligned} &\text{(i)} && p''(x) = -1 \text{ for } 0 < x < 1 \text{ and } p(0) = 0, p'(1) = 0 \\ &\text{(ii)} && p''(x) = -1 \text{ for } 2 < x < 3 \text{ and } p'(2) = 0, p(3) = 1 \\ &\text{(iii)} && p''(x) = 0 \text{ for } 1 < x < 2 \text{ and } p(1) = p(1-), p(2) = p(2+), \end{aligned} \tag{4.2}$$

where $p(1-)$ and $p(2+)$ are defined by (i) and (ii), respectively. Thus we conclude that the problem 4.1 has a well-defined limit as $\epsilon \to 0$ and that the limit is determined by 4.2. The rest of this section is concerned with the rigorous proof of a more general but similar result in 2D.

4.1. An auxiliary result. As mentioned above, the key point in our analysis is to derive a uniform bound of $\| p_\epsilon \|_{H^1(\Omega)}$. Hence we start by proving the following result.

PROPOSITION 4.1. *Suppose f satisfies (2.5), then there exists a constant c_2, independent of $\epsilon \in (0,1)$, such that*

$$\| p_\epsilon \|_{H^1(\Omega)} \le c_2,$$

where p_ϵ is the solution of (3.5).

Proof. We start by showing that the Sobolev norm of $p_\epsilon|_{\Omega_1}$ can be bounded independent of ϵ. By choosing $\psi = p_\epsilon - \overline{p}_{\text{dir}} \in V_\Omega$ in (3.5) it is easy to see that

$$\int_\Omega (\nabla p_\epsilon - \nabla \overline{p}_{\text{dir}}) \cdot [\Lambda_\epsilon (\nabla p_\epsilon - \nabla \overline{p}_{\text{dir}})] \, dx = \\ - \int_\Omega \Lambda_\epsilon^{1/2} (\nabla p_\epsilon - \nabla \overline{p}_{\text{dir}}) \cdot [\Lambda_\epsilon^{1/2} (\nabla \overline{p}_{\text{dir}} - \mathbf{Q})] \, dx$$

$$+\int_{\Omega_1}(p_\epsilon-\overline{p}_{\rm dir})f\;dx-\int_{\Gamma_{\rm neu}}(p_\epsilon-\overline{p}_{\rm dir})g_{\rm neu}\;ds. \tag{4.3}$$

Schwarz's inequality and the assumption (3.4) implies that

$$\begin{aligned}&|-\int_\Omega \Lambda_\epsilon^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})\cdot[\Lambda_\epsilon^{1/2}(\nabla\overline{p}_{\rm dir}-\mathbf{Q})]\;dx|\\ &\qquad\le\sqrt{M}\left[\int_\Omega|(\nabla\overline{p}_{\rm dir}-\mathbf{Q})|^2\;dx\right]^{1/2}\left[\int_\Omega|\Lambda_\epsilon^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})]|^2\;dx\right]^{1/2}.\end{aligned} \tag{4.4}$$

By Schwarz's inequality, Poincaré's inequality (3.7) and (3.3) it follows that

$$|\int_{\Omega_1}(p_\epsilon-\overline{p}_{\rm dir})f\;dx|\le\frac{\|f\|_{L^2(\Omega_1)}c_1}{\sqrt{m}}\left[\int_\Omega|\Lambda_\epsilon^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})|^2\;dx\right]^{1/2}. \tag{4.5}$$

Let $T_{\Omega_1}:H^1(\Omega_1)\to H^{1/2}(\partial\Omega_1)$ be the trace map. Since T_{Ω_1} is a continuous linear operator, and $\Gamma_{\rm neu}\subset\partial\Omega_1$ it is easy to see that

$$|\int_{\Gamma_{\rm neu}}(p_\epsilon-\overline{p}_{\rm dir})g_{\rm neu}\;ds|\le\|g_{\rm neu}\|_{L^2(\Gamma_{\rm neu})}\;\|T_{\Omega_1}\|\;\|p_\epsilon-\overline{p}_{\rm dir}\|_{H^1(\Omega_1)}. \tag{4.6}$$

Consequently, applying Poincaré's inequality (3.7) and (3.3) yields

$$\|p_\epsilon-\overline{p}_{\rm dir}\|_{H^1(\Omega_1)}\le\frac{\sqrt{(1+c_1^2)}}{\sqrt{m}}\left[\int_\Omega|\Lambda_\epsilon^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})]|^2\;dx\right]^{1/2}. \tag{4.7}$$

Thus, from (4.3)-(4.7) and the triangle inequality it follows that

$$\left[\int_\Omega|\Lambda_\epsilon^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})]|^2\;dx\right]^{1/2}\le c_3,$$

where c_3 is a constant independent of ϵ. From (3.2) and the assumption that $\Lambda_\epsilon(x)=\Lambda(x)$ for all $x\in\Omega_1$ we find that

$$\sqrt{m}\left[\int_{\Omega_1}|(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})|^2\right]^{1/2}\le\left[\int_{\Omega_1}|\Lambda^{1/2}(\nabla p_\epsilon-\nabla\overline{p}_{\rm dir})]|^2\;dx\right]^{1/2}\le c_3,$$

and then by Poincaré's inequality (3.7) and the triangle inequality

$$\|p_\epsilon\|_{H^1(\Omega_1)}\le c_4, \tag{4.8}$$

where c_4 is a constant independent of ϵ.

Next, we compare $p_\epsilon|_{\Omega_2}$ with the harmonic extension of $p_\epsilon|_{\partial\Omega_2}$ to Ω_2. To this end, consider the potential equation

$$\Delta u = 0 \qquad \text{in } \Omega_2,$$

with the boundary condition

$$u = T_{\Omega_1}(p_\epsilon)|_{\partial\Omega_2} \qquad \text{on } \partial\Omega_2,$$

where we recall that $\partial\Omega_2 \subset \partial\Omega_1$. Let u be the weak solution of this problem. Then, by a well-known a priori inequality there exist a constant c_5, independent of ϵ, such that

$$\|u\|_{H^1(\Omega_2)} \leq c_5 \|p_\epsilon\|_{H^{1/2}(\partial\Omega_2)}, \tag{4.9}$$

see for instance Hackbusch[13, Ch. 7.3]. Since $u = T_{\Omega_1}(p_\epsilon)|_{\partial\Omega_2}$ on $\partial\Omega_2$ we may choose[2]

$$\psi = \begin{cases} 0 & \text{in } \Omega_1, \\ p_\epsilon - u & \text{in } \Omega_2, \end{cases} \tag{4.10}$$

in (3.5) to obtain

$$\int_{\Omega_2} \nabla(p_\epsilon - u) \cdot [\Lambda_\epsilon(\nabla p_\epsilon - \mathbf{Q})] \, dx = 0.$$

Since $\Lambda_\epsilon(x) = \epsilon\Lambda(x)$ for all $x \in \Omega_2$ we find that

$$\int_{\Omega_2} (\nabla p_\epsilon - \mathbf{Q}) \cdot [\Lambda(\nabla p_\epsilon - \mathbf{Q})] \, dx = \int_{\Omega_2} (\nabla u - \mathbf{Q}) \cdot [\Lambda(\nabla p_\epsilon - \mathbf{Q})] \, dx.$$

Consequently, by (3.2), Schwarz's inequality and (3.3) we get

$$\left(\int_{\Omega_2} |\nabla p_\epsilon - \mathbf{Q}|^2 \, dx \right)^{1/2} \leq \frac{M}{m} \left(\int_{\Omega_2} |\nabla u - \mathbf{Q}|^2 \, dx \right)^{1/2},$$

where we recall from (3.2) that m and M are positive constants not depending on ϵ. The triangle inequality now implies that

$$\left(\int_{\Omega_2} |\nabla p_\epsilon|^2 \, dx \right)^{1/2} \leq \frac{M}{m} \left(\int_{\Omega_2} |\nabla u|^2 \, dx \right)^{1/2} + (1 + M/m) \left(\int_{\Omega_2} |\mathbf{Q}|^2 \, dx \right)^{1/2}. \tag{4.11}$$

[2] Every function $w \in H_0^1(\Omega_2)$ has a canonical extension $\overline{w} \in H_0^1(\Omega) \subset V_\Omega$ obtained by putting $\overline{w} = 0$ in Ω_1 and $\overline{w} = w$ in Ω_2, see for instance Dautray and Lions [7, Ch. IV.4].

Now, from (4.9), the fact that $\partial\Omega_2 \subset \partial\Omega_1$, the boundedness of the trace operator T_{Ω_1} and (4.8) it follows that

$$\left(\int_{\Omega_2} |\nabla u|^2 \, dx\right)^{1/2} \leq c_5 \| p_\epsilon \|_{H^{1/2}(\partial\Omega_2)} \leq c_5 \|T_{\Omega_1}\| \, \| p_\epsilon \|_{H^1(\Omega_1)} \leq c_5 \|T_{\Omega_1}\| c_4.$$

Combining this inequality with (4.11) we find that

$$\left(\int_{\Omega_2} |\nabla p_\epsilon|^2 \, dx\right)^{1/2} \leq \frac{M}{m} c_5 \|T_{\Omega_1}\| c_4 + (1 + M/m) \left(\int_{\Omega_2} |\mathbf{Q}|^2 \, dx\right)^{1/2}$$

This bound together with (4.8) implies that

$$\left(\int_{\Omega} |\nabla p_\epsilon|^2 \, dx.\right)^{1/2} \leq \tilde{c},$$

where $\tilde{c}$ is a constant independent of ϵ. Finally, the desired result follows from this inequality, Poincaré's inequality applied to $p_\epsilon - \overline{p}_{\text{dir}}$ and the triangle inequality. □

4.2. Proof of Theorem 2.1. Subtract (3.6) from (3.5) and use definition (2.3) of Λ_ϵ to obtain

$$\begin{aligned} \text{(4.12)} \quad & \int_{\Omega_1} \nabla\psi \cdot [\Lambda(\nabla p_\epsilon - \mathbf{Q})] \, dx - \int_{\Omega_1} \nabla\varphi \cdot [\Lambda(\nabla p - \mathbf{Q})] \, dx = \\ & - \int_{\Omega_2} \nabla\psi \cdot [\epsilon\Lambda(\nabla p_\epsilon - \mathbf{Q})] \, dx + \int_{\Omega_1} (\psi - \varphi) f \, dx - \int_{\Gamma_{\text{neu}}} (\psi - \varphi) g_{\text{neu}} \, dx, \end{aligned}$$

for all $\psi \in V_\Omega$ and $\varphi \in V_{\Omega_1}$.

Since $p_\epsilon|_{\Gamma_{\text{dir}}} = p|_{\Gamma_{\text{dir}}} = p_{\text{dir}}$ we may choose $\varphi = p_\epsilon|_{\Omega_1} - p \in V_{\Omega_1}$ in (4.12). Next, we want to make an appropriate choice of ψ. To this end, let u be the weak solution of the following potential equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega_2, \\ u &= T_{\Omega_1}(p_\epsilon - p)|_{\partial\Omega_2} \quad \text{on } \partial\Omega_2, \end{aligned}$$

where $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$ is the trace operator. Then, as in the proof of Proposition 4.1 there is a constant c_5, independent of ϵ, such that

$$\text{(4.13)} \qquad \| u \|_{H^1(\Omega_2)} \leq c_5 \| p_\epsilon - p \|_{H^{1/2}(\partial\Omega_2)}.$$

Now let[3]

$$\psi = \begin{cases} p_\epsilon - p & \text{in } \Omega_1 \\ u & \text{in } \Omega_2, \end{cases}$$

which belongs to V_Ω since $u = T_{\Omega_1}(p_\epsilon - p)|_{\partial\Omega_2}$ on $\partial\Omega_2$. With these definitions of φ and ψ in (4.12) we get

$$(4.14) \quad \int_{\Omega_1} (\nabla p_\epsilon - \nabla p) \cdot [\Lambda(\nabla p_\epsilon - \nabla p)]\, dx = -\int_{\Omega_2} \nabla u \cdot [\epsilon\Lambda(\nabla p_\epsilon - \mathbf{Q})]\, dx,$$

where we have used the fact that $\Gamma_{\text{neu}} \subset \partial\Omega \subset \partial\Omega_1$, i.e. $\psi = \varphi$ on Γ_{neu}. The assumption (3.2) implies that

$$(4.15) \qquad \int_{\Omega_1} (\nabla p_\epsilon - \nabla p) \cdot [\Lambda(\nabla p_\epsilon - \nabla p)\, dx \geq m \int_{\Omega_1} |\nabla p_\epsilon - \nabla p|^2\, dx,$$

where we recall that m is independent of ϵ. Next, by Schwarz's inequality for the inner product on $L^2(\Omega_2)$, (3.3), (4.13), the fact that $\partial\Omega_2 \subset \partial\Omega_1$ and the boundedness of the trace operator T_{Ω_1} we get

$$\begin{aligned} \Big|\int_{\Omega_2} \nabla u \cdot [\epsilon\Lambda(\nabla p_\epsilon - \mathbf{Q})]\, dx\Big| &\leq c_5 \| p_\epsilon - p \|_{H^{1/2}(\partial\Omega_2)}\, \epsilon M \left(\int_{\Omega_2} |\nabla p_\epsilon - \mathbf{Q}|^2\, dx \right)^{1/2} \\ (4.16) \qquad &\leq c_5 \|T_{\Omega_1}\|\, \| p_\epsilon - p \|_{H^1(\Omega_1)} \epsilon M \left(\int_{\Omega_2} |\nabla p_\epsilon - \mathbf{Q}|^2\, dx \right)^{1/2}. \end{aligned}$$

Hence, from (4.14)-(4.16), the triangle inequality and Proposition 4.1 we find that

$$\int_{\Omega_1} |\nabla p_\epsilon - \nabla p|^2\, dx \leq \frac{c_5 M \|T_{\Omega_1}\|}{m} \left(c_2 + \left(\int_{\Omega_2} |\mathbf{Q}|^2\, dx \right)^{1/2} \right) \epsilon\, \| p_\epsilon - p \|_{H^1(\Omega_1)}.$$
(4.17)

Combining (4.17) with Poincaré's inequality (3.7) we finally obtain

$$\|p_\epsilon - p\|^2_{H^1(\Omega_1)} \leq c\epsilon\, \| p_\epsilon - p \|_{H^1(\Omega_1)},$$

where $c = c_5 M \|T_{\Omega_1}\|\, (c_2 + \|\mathbf{Q}\|_{(L^2(\Omega))^2})(1 + c_1^2)/m$, and the proof is complete.

[3] Another way to extend $p_\epsilon|_{\Omega_1} - p$ to the entire domain Ω is to appeal to Calderón's extension theorem, see for instance Adams [1, Ch.IV] or Marti [17, Ch.5]. Indeed, by this theorem there exists a bounded linear operator $E : H^1(\Omega_1) \to H^1(\Omega)$ such that $\phi = E(\phi)|_{\Omega_1}$ for all $\phi \in H^1(\Omega_1)$. These properties are sufficient to prove our result. However, we find it more satisfactory to introduce an auxiliary problem since this technique is easy to generalize to the discrete case, cf. Section 5.

4.3. Proof of Theorem 2.2. Since $\mathbf{v} = 0$ on Ω_2, $\Lambda_\epsilon = \Lambda$ on Ω_1, $\Lambda_\epsilon = \epsilon\Lambda$ on Ω_2 and $|\Lambda(x)| \leq M$ for all $x \in \Omega$ we find that

$$\begin{aligned} \int_\Omega |\mathbf{v}_\epsilon - \mathbf{v}|^2 \, dx &= \int_{\Omega_1} |-\Lambda_\epsilon(\nabla p_\epsilon - \mathbf{Q}) + \Lambda(\nabla p - \mathbf{Q})|^2 \, dx + \int_{\Omega_2} |\Lambda_\epsilon(\nabla p_\epsilon - \mathbf{Q})|^2 \, dx \\ &= \int_{\Omega_1} |\Lambda(\nabla p_\epsilon - \nabla p)|^2 \, dx + \int_{\Omega_2} |\epsilon\Lambda(\nabla p_\epsilon - \mathbf{Q}|^2 \, dx \\ &\leq M^2 \int_{\Omega_1} |\nabla p_\epsilon - \nabla p|^2 \, dx + \epsilon^2 M^2 \int_{\Omega_2} |\nabla p_\epsilon - \mathbf{Q}|^2 \, dx. \end{aligned}$$

Hence, from Theorem 2.1, the triangle inequality and Proposition 4.1 it follows that

$$\int_\Omega |\mathbf{v}_\epsilon - \mathbf{v}|^2 \, dx \leq M^2 c^2 \epsilon^2 + M^2 (c_2 + \| \mathbf{Q} \|_{(L^2(\Omega))^2})^2 \epsilon^2,$$

which finishes the proof.

5. The convergence results for discrete problems. We proved above that the approximation described in Section 2 converges in the sense that the error is of order ϵ measured in the proper norms. Obviously, equations of the form (1.1) are solved numerically for all practical purposes. Therefore, it is important to verify that similar results hold in the discrete case. In this section we will consider finite element discretizations of (2.4), (2.6) and (2.8), (2.9) and prove that exactly the same convergence properties hold for the finite element solutions, i.e. the error is again $O(\epsilon)$ in proper norms. This result will be complemented by a numerical experiment which shows that $O(\epsilon)$ is an optimal result.

Before we start deriving the convergence result in the discrete case, let us remark that a rough result can easily be derived by using the triangle inequality, the convergence properties of the finite element method, and the result obtained above. The problem with this straight forward approach is that it leads to an error of the form $O(h^\alpha) + O(\epsilon)$ rather than just $O(\epsilon)$. Obviously, for a convergent finite element method the term $O(h^\alpha)$, $\alpha > 0$, stems from the discretization error of the finite element method, see for instance Hackbusch [13] or Marti [17]. This rough result indicates an error of order $O(h^\alpha)$ as ϵ tends to zero. However, by following closely the steps in the proofs above, we can prove that the error, also in the discrete case, is of order ϵ.

We start by introducing some notation and by formulating finite element methods for the problems (3.5) and (3.6) above. Let $\{N_1, \ldots, N_q\}$ be a set of functions such that $N_i|_{\Gamma_{\mathrm{dir}}} = 0$ for $i = 1, \ldots, q$, and define

$$\begin{aligned} V_{\Omega,h} &= \mathrm{span}\{\mathrm{N}_1, \ldots, \mathrm{N}_\mathrm{q}\}, \\ V_{\Omega_1,h} &= \mathrm{span}\{\mathrm{N}_1|_{\Omega_1}, \ldots, \mathrm{N}_\mathrm{q}|_{\Omega_1}\}, \\ V_{\Omega_2,h} &= \mathrm{span}\{N_1|_{\Omega_2}, \ldots, N_q|_{\Omega_2}\}. \end{aligned}$$

Here, the subscript $h \in I$, where I is some subset of $\mathbb{R}_+$, is used to distinguish the finite dimensional entities from the corresponding symbols used in Section 2-Section 4. Typically, h is the global mesh parameter for a grid defined on Ω. Details on how to construct appropriate finite dimensional spaces $V_{\Omega,h}$, $V_{\Omega_1,h}$ and $V_{\Omega_2,h}$ can be found in Ciarlet [6] and Marti [17, Ch.8]

In order to formulate the convergence results for the finite dimensional problems we must introduce some assumptions on the sets $V_{\Omega_1,h}$, $V_{\Omega_2,h}$, $V_{\Omega,h}$. To this end, let $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$ and $T_{\Omega_2} : H^1(\Omega_2) \to H^{1/2}(\partial\Omega_2)$ be the trace operators and introduce the sets

$$\begin{aligned} G_{\Omega_1,h} &= \{T_{\Omega_1}(\psi)|_{\partial\Omega_2};\ \psi \in V_{\Omega_1,h}\}, \\ G_{\Omega_2,h} &= \{T_{\Omega_2}(\psi);\ \psi \in V_{\Omega_2,h}\}. \end{aligned}$$

We will prove the convergence results in the discrete case under the following assumptions:
For every $h \in I$;

1. $N_i \in V_\Omega$ for $i = 1, \ldots, q$ (which implies that $V_{\Omega,h} \subset V_\Omega$, $V_{\Omega_1,h} \subset V_{\Omega_1}$ and $V_{\Omega_2,h} \subset H^1(\Omega_2)$).
2. $G_{\Omega_1,h} = G_{\Omega_2,h}$.
3. If $w \in V_{\Omega_1,h}$ and $u \in V_{\Omega_2,h}$ satisfies $T_{\Omega_1}(w)|_{\partial\Omega_2} = T_{\Omega_2}(u)$ then the function

$$\psi = \begin{cases} w & \text{on } \Omega_1 \\ u & \text{on } \Omega_2 \end{cases}$$

   belongs to $V_{\Omega,h}$.
4. For every $w \in G_{\Omega_2,h}$ the following discrete form of the potential equation has a unique solution: Find $u_h \in V_{\Omega_2,h}$ such that $u_h|_{\partial\Omega_2} = w$ and

$$\int_{\Omega_2} \nabla u_h \cdot \nabla v \; dx = 0 \quad \text{for all } v \in V_{\Omega_2,h} \cap H_0^1(\Omega_2).$$

Furthermore, there exists a constant c_6 independent of ϵ and h such that

$$\| u_h \|_{H^1(\Omega_2)} \le c_6 \| w \|_{H^{1/2}(\partial\Omega_2)}.$$

As mentioned above, we want to derive discrete versions of the results discussed in Section 4 by following the same steps as in the continuous case. In order to do this, the conditions stated above have to be satisfied. In particular, Condition 1 assures that the finite dimensional spaces are subspaces of the appropriate Sobolev spaces, thus making the inequalities of Poincaré and Schwarz applicable for the analysis of the discrete problems. Next, the conditions 2, 3 and 4 allow us to extend functions defined on Ω_1 to functions defined on the entire domain Ω. Furthermore, Condition 4 assures that we have a discrete harmonic extension of functions defined on $\partial\Omega_2$ to a function defined on Ω_2. The properties of the extended function is completely analogous to the continuous extension used in the proofs of Proposition 4.1 and Theorem 2.1 above.

The conditions 1-4 are typically satisfied for simple geometries as e.g. in our prototype problem depicted in Figure 1. Generally, conditions 1-3 are easily checked for a given geometry, whereas Condition 4 is harder to verify. This issue is carefully discussed in the paper by Bramble, Pasciak and Schatz [2], [3].

The Ritz-Galerkin discretization of (3.5) is defined as follows: Find $p_{\epsilon,h} \in \overline{p}_{\text{dir}} + V_{\Omega,h}$ such that

$$\int_{\Omega} \nabla\psi \cdot [\Lambda_\epsilon(\nabla p_{\epsilon,h} - \mathbf{Q})]\, dx = \int_{\Omega_1} f\psi\, dx - \int_{\Gamma_{\text{neu}}} \psi g_{\text{neu}}\, ds \tag{5.1}$$

for all $\psi \in V_{\Omega,h}$. Similarly, the discrete approximation of (3.6) is: Find $p_h \in \overline{p}_{\text{dir},\Omega_1} + V_{\Omega_1,h}$ such that

$$\int_{\Omega_1} \nabla\varphi \cdot [\Lambda(\nabla p_h - \mathbf{Q})]\, dx = \int_{\Omega_1} f\varphi\, dx - \int_{\Gamma_{\text{neu}}} \varphi g_{\text{neu}}\, ds \tag{5.2}$$

for all $\varphi \in V_{\Omega_1,h}$.

Now we have made the necessary preparations to formulate our convergence results for the discrete problems (5.1) and (5.2). Since the proofs in the discrete case are similar to those in the continuous case, we leave some details of the proofs out, and try to focus on where the assumptions 1-4 are used.

PROPOSITION 5.1. *Assume that $V_{\Omega,h}$, $V_{\Omega_1,h}$ and $V_{\Omega_2,h}$ satisfies conditions 1-4 and that f satisfies (2.5). Furthermore, suppose that the problem (5.1) has a unique* *solution $p_{\epsilon,h}$ for every $\epsilon \in (0,1)$ and every $h \in I$, then there exists a constant c_7, independent of $\epsilon \in (0,1)$ and $h \in I$, such that*

$$\| p_{\epsilon,h} \|_{H^1(\Omega)} \leq c_7.$$

Proof. Choose $\psi = p_{\epsilon,h} - \overline{p}_{\rm dir} \in V_{\Omega,h}$ in (5.1). Then, from condition 1 and the fact that $(p_{\epsilon,h} - \overline{p}_{\rm dir})|_{\Omega_1} \in V_{\Omega_1,h}$, it follows exactly as in the proof of Proposition 4.1 that

$$\| p_{\epsilon,h} \|_{H^1(\Omega_1)} \leq c_4, \tag{5.3}$$

where c_4 is a constant independent of ϵ and h, cf. equations (4.3)-(4.8).

Now, consider the following problem: Find $\tau_h \in V_{\Omega_2,h}$ such that $\tau_h|_{\partial\Omega_2} = T_{\Omega_1}(p_{\epsilon,h} - \overline{p}_{\rm dir})|_{\partial\Omega_2}$ and

$$\int_{\Omega_2} \nabla\tau_h \cdot \nabla v \; dx = 0 \quad \text{for all } v \in V_{\Omega_2,h} \cap H_0^1(\Omega_2).$$

From condition 2 it is evident that $T_{\Omega_1}(p_{\epsilon,h} - \overline{p}_{\rm dir})|_{\partial\Omega_2} \in G_{\Omega_2,h}$ and then from condition 4 we find that

$$\| \tau_h \|_{H^1(\Omega_2)} \leq c_6 \| p_{\epsilon,h} - \overline{p}_{\rm dir} \|_{H^{1/2}(\partial\Omega_2)},$$

where c_6 is a constant independent of ϵ and h.

Let $u_h = \tau_h + \overline{p}_{\rm dir}|_{\Omega_2}$, and observe that

$$\begin{aligned} \| u_h \|_{H^1(\Omega_2)} &\leq \| \tau_h \|_{H^1(\Omega_2)} + \| \overline{p}_{\rm dir} \|_{H^1(\Omega_2)} \\ &\leq c_6 \| p_{\epsilon,h} - \overline{p}_{\rm dir} \|_{H^{1/2}(\partial\Omega_2)} + \| \overline{p}_{\rm dir} \|_{H^1(\Omega_2)} \\ &\leq c_6 \|T_{\Omega_1}\| \; \| p_{\epsilon,h} \|_{H^1(\Omega_1)} + (1 + c_6 \|T_{\Omega_1}\|) \; \| \overline{p}_{\rm dir} \|_{H^1(\Omega)}, \end{aligned} \tag{5.4}$$

where we have used the triangle inequality and the boundedness of the trace operator T_{Ω_1} (recall that $\partial\Omega_2 \subset \partial\Omega_1$).

Next, by assumption 3 we may choose

$$\psi = \begin{cases} 0 & \text{in } \Omega_1 \\ p_{\epsilon,h} - u_h & \text{in } \Omega_2 \end{cases},$$

in (5.1) to obtain

$$\left(\int_{\Omega_2} |\nabla p_{\epsilon,h}|^2\, dx\right)^{1/2} \leq \frac{M}{m}\left(\int_{\Omega_2} |\nabla u_h|^2\, dx\right)^{1/2} + (1+M/m)\left(\int_{\Omega_2} |\mathbf{Q}|^2\, dx\right)^{1/2}, \tag{5.5}$$

cf. equations (4.10) to (4.11).

The desired result now follows from (5.3)-(5.5), cf. the proof of Proposition 4.1.

□

THEOREM 5.2. *Assume that conditions 1-4 on the spaces $V_{\Omega,h}$, $V_{\Omega_1,h}$ and $V_{\Omega_2,h}$ hold. Suppose f satisfy (2.5) and that the problems (5.1) and (5.2) have unique solutions $p_{\epsilon,h}$ and p_h, respectively. Then there exists a constant c, independent of ϵ and h, such that*

$$\| p_{\epsilon,h} - p_h \|_{H^1(\Omega_1)} \leq c\epsilon.$$

Proof. From the definition of $V_{\Omega_1,h}$ and $V_{\Omega,h}$ we may choose $\varphi = p_{\epsilon,h}|_{\Omega_1} - p_h \in V_{\Omega_1,h}$ in (5.2). Next, consider the following problem: Find $u_h \in V_{\Omega_2,h}$ such that $u_h|_{\partial\Omega_2} = T_{\Omega_1}(p_{\epsilon,h} - p_h)|_{\partial\Omega_2}$ and

$$\int_{\Omega_2} \nabla u_h \cdot \nabla v\, dx = 0 \qquad \text{for all } v \in V_{\Omega_2,h} \cap H_0^1(\Omega_2).$$

Due to conditions 2 and 4 this problem has a unique solution satisfying

$$\| u_h \|_{H^1(\Omega_2)} \leq c_6 \| p_{\epsilon,h} - p_h \|_{H^{1/2}(\partial\Omega_2)}. \tag{5.6}$$

Now, by condition 3 we can put

$$\psi = \begin{cases} p_{\epsilon,h} - p_h & \text{in } \Omega_1 \\ u_h & \text{in } \Omega_2 \end{cases},$$

in (5.1). With these choices of φ and ψ in (5.2) and (5.1), respectively, we obtain by subtracting (5.2) from (5.1)

$$\int_{\Omega_1} (\nabla p_{\epsilon,h} - \nabla p_h) \cdot [\Lambda(\nabla p_{\epsilon,h} - \nabla p_h)]\, dx = -\int_{\Omega_2} \nabla u_h \cdot [\epsilon\Lambda(\nabla p_{\epsilon,h} - \mathbf{Q})]\, dx,$$

cf. equation (4.14). Now, the result follows exactly as in the proof of Theorem 2.1 by applying condition 1, inequality (5.6) and Proposition 5.1, cf. inequalities (4.15)-(4.17). □

As in the continuous case, the associated velocity fields are defined by

$$
\begin{aligned}
\mathbf{v}_{\epsilon,\mathbf{h}} &= -\Lambda_\epsilon(\nabla p_{\epsilon,h} - \mathbf{Q}) \quad \text{on } \Omega, \\
\mathbf{v}_{\mathbf{h}} &= \begin{cases} -\Lambda(\nabla p_h - \mathbf{Q}) & \text{on } \Omega_1 \\ 0 & \text{on } \Omega_2, \end{cases}
\end{aligned} \tag{5.7}
$$

where $p_{\epsilon,h}$ and p_h are the solutions of (5.1) and (5.2), respectively.

THEOREM 5.3. *Assume that conditions 1-4 on the spaces $V_{\Omega,h}$, $V_{\Omega_1,h}$ and $V_{\Omega_2,h}$ hold, and that f satisfy (2.5). Suppose that the problems (5.1) and (5.2) have unique solutions $p_{\epsilon,h}$ and p_h, respectively, and let the velocity vectors $\mathbf{v}_{\epsilon,\mathbf{h}}$ and $\mathbf{v}_{\mathbf{h}}$ be defined by (5.7). Then the following inequality hold*

$$
\| \mathbf{v}_{\epsilon,\mathbf{h}} - \mathbf{v}_{\mathbf{h}} \|_{(L^2(\Omega))^2} \leq c\epsilon,
$$

where c is a constant independent of ϵ and h.

The proof of this Theorem is analogous to the proof of Theorem 2.2 and therefore omitted, cf. Section 4.

A numerical experiment. Now we turn our attention to a simple test problem. Let the domain specifications be $\Omega = (0,3) \times (0,3)$ and $\Omega_2 = (1,2) \times (1,2)$. Moreover, the boundary segments are defined by

$$
\begin{aligned}
\Gamma_{\text{dir}} &= \left\{(x,y) \in \mathbb{R}^2;\ x = 3 \text{ and } 2.75 \leq y \leq 3\right\}, \\
\Gamma_{\text{neu}} &= \partial\Omega \setminus \Gamma_{\text{dir}},
\end{aligned}
$$

see Section 2. Hence, the geometry in our test problem is similar to the domain shown in Figure 1. Furthermore, we put $f = 0$, $\mathbf{Q} = 0$, $p_{\text{dir}} = 0$, and

$$
g_{\text{neu}}(x,y) = \begin{cases} 1 & \text{for } x = 0 \text{ and } 0 \leq y \leq 0.25 \\ 0 & \text{elsewhere.} \end{cases}
$$

The physical interpretation of these functions for our prototype of the pressure equation (2.4) can be found in Section 2. Finally, the mobility tensor Λ is chosen to be the identity matrix, and Λ_ϵ is given by (2.3).

The experiment described in this section has been carried out for bilinear shape functions on quadrilateral elements, where the values of $p_{\epsilon,h}$ or p_h corresponding to the four vertices of each element represent the degrees of freedom. That is, for a mesh parameter h such that $1/h$ and $3/h$ are integers a uniform grid consisting of quadratic elements with sides of length h is defined on Ω. The resulting linear system of equations has been solved by the preconditioned conjugate gradient (CG) method, cf. e.g. Meijerink and van der Vorst [18]. All computations have been carried out in double precision on HP 9000/735 workstations. The implementations are based on the C++ class library Diffpack, which is under development at SINTEF and the University of Oslo, see Langtangen [16] and [9].

In this case, conditions 1-4 on $V_{\Omega_1,h}$, $V_{\Omega_2,h}$ and $V_{\Omega,h}$ stated above, hold. Hence, Theorems 5.2 and 5.3 apply to our test problem. The problem (5.1) has been solved for $\epsilon = 2^{-n}$, $n = 0, \ldots, 16$. For every value of ϵ the solution $p_{\epsilon,h}$ of (5.1) has been compared with the solution p_h of (5.2). The rate of convergence is computed by comparing the results of two successive values of ϵ and assume that the difference $\|p_{\epsilon,h} - p_h\|_{H^1(\Omega_1)}$ has the form $c\epsilon^\alpha$ where c is a constant and α is the rate. Table 1 shows the numerical results computed with mesh-size $h = 1/60$. The estimated rate of convergence, with respect to ϵ, clearly tends towards 1.0 as ϵ goes to zero. This is in agreement with Theorem 5.2.

ϵ	$\|p_{\epsilon,h} - p_h\|_{H^1(\Omega_1)}$	Rate
1.0000000	0.2058372	-
0.5000000	0.1481394	0.4745478
0.2500000	0.0963269	0.6209443
0.1250000	0.0571385	0.7534755
0.0625000	0.0316098	0.8540916
0.0312500	0.0167119	0.9194984
0.0156250	0.0086056	0.9575259
0.0078125	0.0043684	0.9781557
0.0039062	0.0022011	0.9889189
0.0019531	0.0011048	0.9944188
0.0009766	0.0005535	0.9971991
0.0004883	0.0002770	0.9985970
0.0002441	0.0001386	0.9992978
0.0001221	0.0000693	0.9996488
0.0000610	0.0000347	0.9998243
0.0000305	0.0000173	0.9999122
0.0000153	0.0000087	0.9999561

TABLE 1

The table shows the numerical results computed with mesh-size $h = 1/60$ for our test problem.

Finally, in Figure 2 we have plotted the velocity fields $\mathbf{v}_{\epsilon,\mathbf{h}}$ for $\epsilon = 1/2,\ 1/4,\ 1/16$ and $\mathbf{v}_\mathbf{h}$, where $\mathbf{v}_{\epsilon,\mathbf{h}}$ and $\mathbf{v}_\mathbf{h}$ are defined in (5.7). We observe from the figure, that the velocity field $\mathbf{v}_{\epsilon,\mathbf{h}}$ converges towards $\mathbf{v}_\mathbf{h}$ as ϵ goes to zero.

6. Concluding remarks. A well-known technique for simplifying the pressure and velocity computations in models arising in reservoir simulation and metal casting has been analyzed. In the simplification procedures, the domain is changed in order to obtain faster solution methods. In the case of oil recovery, areas of low mobility are ignored, by removing these parts from the solution domain, and the problem is solved on the remaining part. Contrary, in models of metal casting, solid areas in the mushy zone, where the rate of flow is equal to zero, are replaced by low mobility areas. In this paper, these techniques have been analyzed for a prototype of an elliptic pressure equation. Analytical estimates have been derived that bound the errors in the pressure and velocity, due to changing the domain, in terms of the order of mobility in the problem areas. Finally, the theoretical work was complemented by a numerical experiment which showed that the estimates are sharp.

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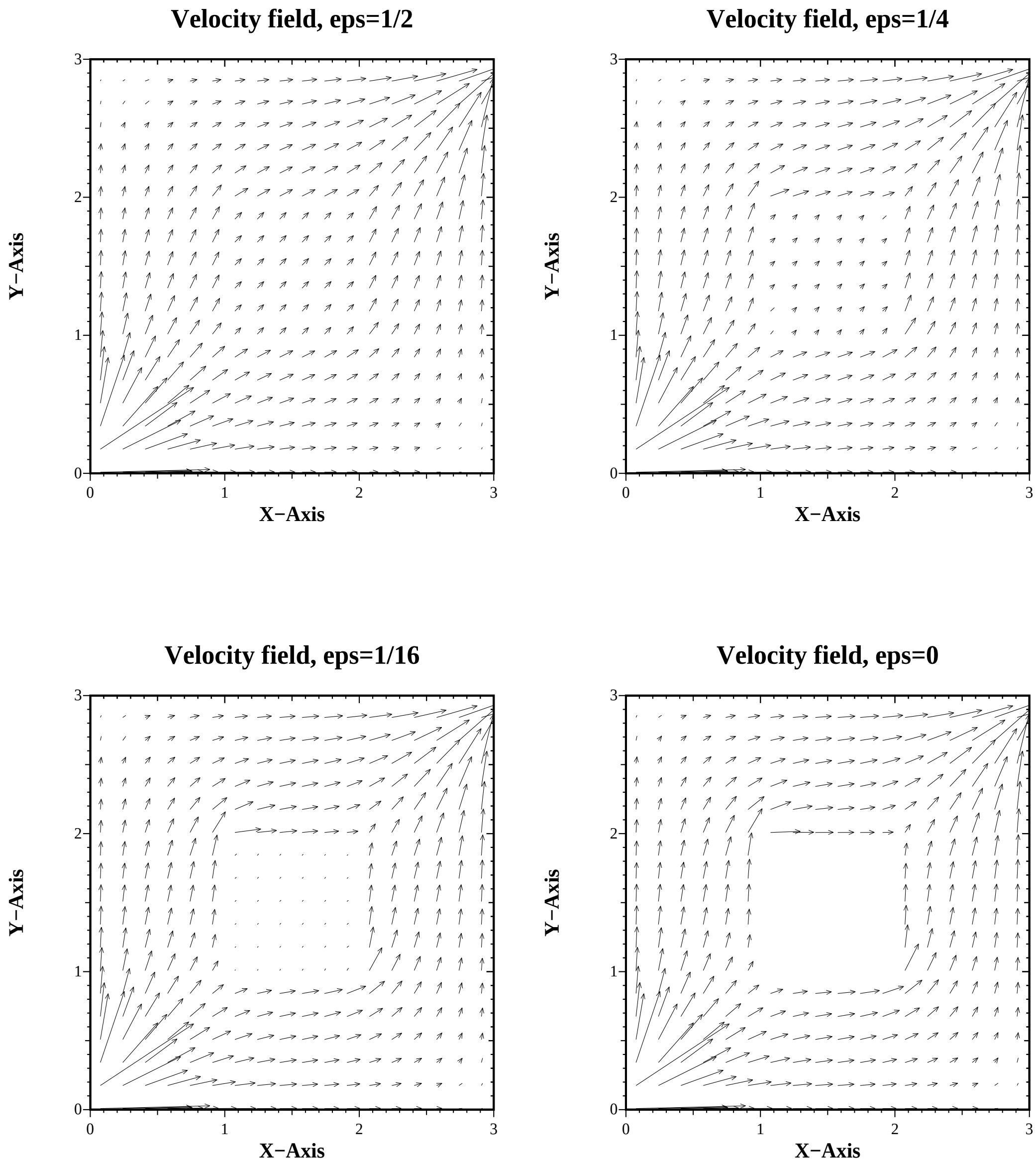

FIG. 2. *The velocity fields* $\mathbf{v}_{\epsilon,\mathbf{h}}$ *for* $\epsilon = 1/2,\ 1/4,\ 1/16$ *and* $\mathbf{v}_{\mathbf{h}}$ *marked with* $\epsilon = 0$ *for our test problem. Here, the mesh size is* $h = 1/60$.