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On the Stability of Solutions of the Pressure Equation with Respect to Perturbations in the Mobility Tensor

In mathematical models of oil recovery, one usually has to solve a pressure equation which is derived form Darcy's law and conservation of mass. Because of the difficulties connected to the measurements of physical parameters such as the permeability of the rock, it is of importance to study the stability of the pressure and the velocity field with respect to perturbations in the mobility tensor. This entity gathers some of the most important physical parameters in models of oil recovery and is a vital parameter in connection to simulations. We present some estimates showing stability in suitable norms.

1. Foundations

In this paper we will concentrate on the elliptic equation

$$\nabla \cdot [\Lambda (\nabla p - \rho g \nabla D)] + \frac{q}{\rho} = 0 \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{1}$$

Here p represents the unknown fluid pressure and $\Lambda : \Omega \to \mathbb{R}^{2\times 2}$ is a second order mobility tensor incorporating various physical parameters as described above. Moreover, the function D denotes the depth of the reservoir measured in the direction of gravity, while g is the gravitational constant and ρ is the fluid density. Throughout this paper we will assume that ρ, g and ∇D are constant over the domain Ω. The boundary $\partial\Omega$ can be divided into two disjoint segments Γ_1 and Γ_2. The pressure equation (1) is then subject to the boundary conditions

$$\begin{aligned} p &= g_1 \qquad \text{on } \Gamma_1, \\ \mathbf{v} \cdot \mathbf{n} &= g_2 \qquad \text{on } \Gamma_2, \end{aligned} \tag{2}$$

i.e. Dirichlet condition on Γ_1 and Neumann condtion on Γ_2. In (2) $\mathbf{n}$ denotes the outward directed normal vector of unit length and $\mathbf{v}$ is the Darcy velocity defined as

$$\mathbf{v} = -\Lambda (\nabla p - \rho g \nabla D).$$

The equations (1)-(2) may be taken as a model of incompressible flow in a heterogeneous reservoir. For further details on such models we refer to Aziz and Settari [1] or Peaceman [6].

We want to study perturbations in the mobility tensor Λ. In order to formulate our results we introduce the following set of mobility tensors

$$A_{m,M} = \left\{ \Lambda : \Omega \to \mathbb{R}^{2\times 2};\ \Lambda = \Lambda^T \text{and } m \leq \frac{\mathbf{z}^T \Lambda(\mathbf{x}) \mathbf{z}}{|\mathbf{z}|^2} \leq M \text{ for all } \mathbf{z} \in \mathbb{R}^2 \setminus \{0\} \text{ and } \mathbf{x} \in \Omega \right\},$$

i.e. the members of $A_{m,M}$ are uniformly positive definite matrices. In the sequel we will assume that the domain Ω and the functions q, g_1 and g_2 are sufficiently smooth such that the weak formulation of the problem (1)-(2) has a unique solution $p \in H^1(\Omega)$ for every $\Lambda \in A_{m,M}$. Details on the necessary regularity assumptions can, for instance, bee found in Gilbarg and Trudinger [3] or Hackbusch [4].

2. Stability analysis

In this section we will study perturbations in the mobility tensor Λ, measured by suitable norms. First, let us introduce the mappings

$$\Psi : A_{m,M} \to H^1(\Omega),$$
$$\Upsilon : A_{m,M} \to (L^2(\Omega))^2,$$

where $\Psi(\Lambda)$ is the weak solution of the problem (1)-(2) corresponding to $\Lambda \in A_{m,M}$ and $\Upsilon(\Lambda) = -\Lambda(\nabla\Psi(\Lambda) - \rho g \nabla D)$

(i.e. the Darcy velocity $\mathbf{v}$ associated with the solution of (1)-(2)). Moreover, on $(L^2(\Omega))^2$ we introduce the norm

$$\|\mathbf{w}\|_{(L^2(\Omega))^2} = (\|w_1\|^2_{L^2(\Omega)} + \|w_2\|^2_{L^2(\Omega)})^{1/2} \quad \text{for } \mathbf{w} = (w_1, w_2) \in (L^2(\Omega))^2.$$

In [2] the following result is proved:

Theorem 1. *Let $p_1 = \Psi(\Lambda^{(1)})$, $p_2 = \Psi(\Lambda^{(2)})$, $\mathbf{v}_1 = \Upsilon(\Lambda^{(1)})$ and $\mathbf{v}_2 = \Upsilon(\Lambda^{(2)})$ where $\Lambda^{(1)}, \Lambda^{(2)} \in A_{m,M}$. There exists constants $c_1, c_2 \in \mathbb{R}_+$ (independent of $\Lambda^{(1)}$ and $\Lambda^{(2)}$, but depending on m and M) such that*

$$\|p_1 - p_2\|_{H^1(\Omega)} \leq c_1 \|\Lambda^{(1)} - \Lambda^{(2)}\|_{L^\infty(\Omega)}$$

and

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_{(L^2(\Omega))^2} \leq c_2 \|\Lambda^{(1)} - \Lambda^{(2)}\|_{L^\infty(\Omega)}.$$

That is, the pressure p and the Darcy velocity $\mathbf{v}$ are uniformly continuous with respect to perturbations in the mobility tensor Λ, measured in the L^∞-norm. This is important in numerical computations, since computers work on finite sets.

However, in reservoir simulation it may be difficult to quantify the mobilities by domain. Typically, Λ could be piecewise constant, thus representing different reservoir layers. When mobilities are changed by geometry, the L^∞-norm is not appropriate for measuring the perturbations, since it will not catch variations in the layer boundaries. Instead we will use the L^1-norm, which incorporates geometric information through integration. Now we introduce the metric $d(\cdot,\cdot)$ on $A_{m,M}$, defined by

$$d(\Lambda^{(1)}, \Lambda^{(2)}) = \|\Lambda^{(1)} - \Lambda^{(2)}\|_{L^1(\Omega)} = \int_\Omega |\Lambda^{(1)} - \Lambda^{(2)}|\, dx.$$

With this notation at hand, our second result is:

Theorem 2. *Let $A_{m,M}$, $H^1(\Omega)$ and $(L^2(\Omega))^2$ be equipped with the $d(\cdot,\cdot)$ metric, the $\|\cdot\|_{H^1(\Omega)}$ norm and the $\|\cdot\|_{(L^2(\Omega))^2}$ norm, respectively. Then the mappings Ψ and Υ are continuous.*

The proof of this theorem can be found in [2]. Hence, the problem (1)-(2) is stable with the topology induced by the $d(\cdot,\cdot)$-metric on $A_{m,M}$. Unfortunately, we have not been able to show that Ψ or Υ are uniformly continuous. Actually, this problem has been studied through a series of numerical experiments. So far, we have not reached any final conclusion.

Finally, we would like to mention that further results on perturbations in the mobility tensor and related domain modification procedures can be found in [2] and [5].

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3. References

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